M1 MAS
Statistics

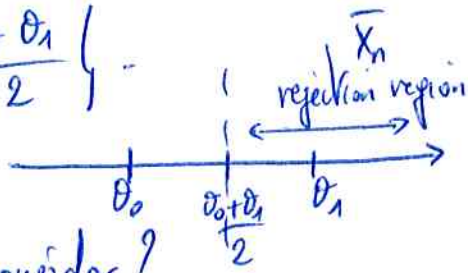
EXERCICES ON STATISTICAL TESTS

Exercise 1: cf course

Exercise 2 Let X be a n-sample $X=(X_1,\dots,X_n)$ with $X_i \sim \mathcal{N}(\theta,1)$.

We know that $\theta \in \Theta = \{\theta_0 ; \theta_1\}$, with $\theta_0 < \theta_1$.

We want to test: $(H_0): \theta=\theta_0$ against $(H_1): \theta=\theta_1$, and consider the rejection region $R(X) = \left\{ \bar{X}_n \geq \frac{\theta_0+\theta_1}{2} \right\}$.



1. Compute the test level α.
2. For fixed α, which value of n should we consider?
3. What is the power of the test?

A. 1. By definition for this unilateral test,

$$\alpha = \mathbb{P}_{H_0}\Big(\underbrace{H_1}_{\text{reject } H_0}\Big) = \mathbb{P}_{H_0}\big(X \in R(X)\big) = \mathbb{P}_{H_0}\left(\bar{X}_n \geq \frac{\theta_0+\theta_1}{2}\right)$$

We know that: → under (H_0) : $\bar{X}_n \sim \mathcal{N}\left(\theta_0, \frac{1}{n}\right)$

→ under (H_1) : $\bar{X}_n \sim \mathcal{N}\left(\theta_1, \frac{1}{n}\right)$

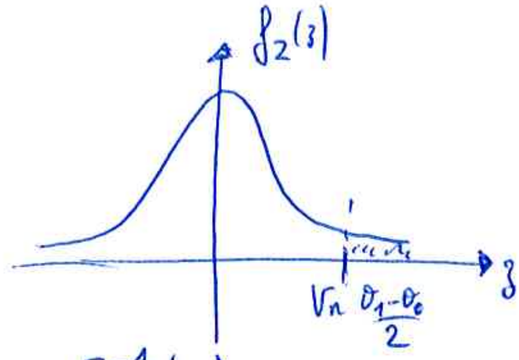
Therefore $\alpha = \mathbb{P}\left(\bar{X}_n - \theta_0 \geq \frac{\theta_0+\theta_1}{2} - \theta_0\right) = \mathbb{P}\left(\bar{X}_n - \theta_0 \geq \frac{\theta_1-\theta_0}{2}\right)$

$$= \mathbb{P}\left(\frac{\bar{X}_n-\theta_0}{\sqrt{\frac{1}{n}}} \geq \frac{(\theta_1-\theta_0)/2}{\sqrt{\frac{1}{n}}}\right) = \mathbb{P}\left(Z \geq \sqrt{n}\,\frac{(\theta_1-\theta_0)}{2}\right)$$

$\sim Z \sim \mathcal{N}(0,1)$

$$= 1 - F_Z\left(\sqrt{n}\,\frac{\theta_1-\theta_0}{2}\right)$$

Then $\alpha = \mathbb{P}\left(Z \leq \sqrt{n}\,\frac{\theta_0-\theta_1}{2}\right)$



② - We know that

$$\alpha = F_Z\left(\sqrt{n}\,\frac{\theta_0-\theta_1}{2}\right) \iff \sqrt{n}\,\frac{\theta_0-\theta_1}{2} = F_Z^{-1}(\alpha)$$

$$\iff \sqrt{n} = \frac{2F_Z^{-1}(\alpha)}{\theta_0-\theta_1} \iff n = \frac{4}{(\theta_0-\theta_1)^2}\left(F_Z^{-1}(\alpha)\right)^2.$$

③ - By definition, we look at

$$\beta = \mathbb{P}_{H_1}(\text{reject } H_0) = \mathbb{P}_{H_1}\left(\bar{X}_n \geq \frac{\theta_0+\theta_1}{2}\right) = \mathbb{P}_{H_1}\left(\frac{\bar{X}_n-\theta_1}{\sqrt{\frac{1}{n}}} \geq \frac{\frac{\theta_0+\theta_1}{2}-\theta_1}{\sqrt{\frac{1}{n}}}\right)$$

$$= \mathbb{P}\left(Z \geq \sqrt{n}\,\frac{\theta_0-\theta_1}{2}\right) = 1-\alpha.$$

Exercice 3 Let X be a n-sample with $X_i \sim \mathcal{P}(\theta)$, $\theta \in \Theta = \{1;2\}$.

We consider the statistical test: $H_0: \theta = 1$ VS $H_1: \theta = 2$.

The rejection region is such that $R(X) = \{\bar{X}_n > 3\}$. If we want a test level equal to 5%, what would be the size n? Compute then the power of the test.

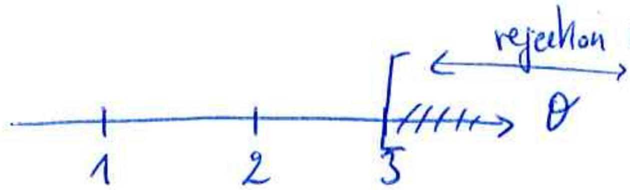
$5\% = \alpha = \mathbb{P}_{H_0}(\text{reject } H_0) = \mathbb{P}_{H_0}(\bar{X}_n > 3)$, knowing that

→ under H_0 : $X_i \sim \mathcal{P}(1) \Rightarrow \sum_{i=1}^n X_i = n\bar{X}_n \sim \mathcal{P}(n)$

→ under H_1 : $X_i \sim \mathcal{P}(2) \Rightarrow \sum_{i=1}^n X_i = n\bar{X}_n \sim \mathcal{P}(2n)$

Remark: Here we know the exact distribution of the test statistic under H_0: these are exact tests! Sometimes we use approximations ... like with asymptotic confidence intervals...

Then,



↳ estimator of the parameter of a Poisson.

(2)

$$0{,}05 = \alpha = \mathbb{P}_{H_0}\Big(\underbrace{n\bar{X}_n}_{\sim Z_1 \sim \mathcal{P}(n)} > 3n\Big) = 1 - F_{Z_1}(3n) = 1 - \sum_{i=0}^{3n} e^{-n}\frac{n^i}{i!}$$

Hence $\displaystyle e^{-n}\sum_{i=0}^{3n}\frac{n^i}{i!} = 0{,}95 \Leftrightarrow \dots$

- For the power of the test, we use once again the definition:

$$\beta = \mathbb{P}_{H_1}(\text{reject } H_0) = \mathbb{P}_{H_1}(\bar{X}_n > 3) = \mathbb{P}_{H_1}(n\bar{X}_n > 3n) \text{ with}$$

$Z_2 = n\bar{X}_n \sim \mathcal{P}(2n)$. Therefore $\beta = \mathbb{P}(Z_2 > 3n) = 1 - F_{Z_2}(3n)$

$$= 1 - \sum_{i=0}^{3n} e^{-2n}\frac{(2n)^i}{i!} = \dots$$

Exercice 4 Let $X = (X_1, \dots, X_n)$, $X_i \overset{iid}{\sim} \mathcal{U}([0,\theta])$; $\hat{\theta}_n = \max(X_1, \dots, X_n)$.

(1) - $$\mathbb{P}\left(\frac{\hat{\theta}_n}{\theta} \le x\right) = \mathbb{P}\left(\frac{\max(X_1,\dots,X_n)}{\theta} \le x\right) = \mathbb{P}\left(\frac{X_1}{\theta} \le x, \dots, \frac{X_n}{\theta} \le x\right)$$

$$= \prod_{i=1}^{n}\mathbb{P}\left(\frac{X_i}{\theta} \le x\right) = \left(\mathbb{P}\left(\frac{X_i}{\theta} \le x\right)\right)^n \text{ since the } X_i \text{ are iid.}$$

Furthermore, we know that $X_i \sim \mathcal{U}([0,\theta]) \Rightarrow \frac{X_i}{\theta} \sim \mathcal{U}([0,1])$.

Then it follows immediately that:

$$\mathbb{P}\left(\frac{\hat{\theta}_n}{\theta} \le x\right) = \left(F_{U_i}(x)\right)^n = x^n. \Rightarrow \text{This distribution is independent from } \theta.$$

② - We want to build a statistical test for:

$$H_0 : \theta = 1 \quad \text{against} \quad H_1 : \theta \neq 1, \text{ with level } \alpha.$$

We follow the steps given in the course to build the test:

- the null and alternative hypothesis were already defined.
- the level α is set.
- We have to choose a test statistic: this test statistic should have a known distribution under H_0, independent from the parameter to test (use a "pivot").

→ first, $\hat{\theta}_n = \max(X_1, \dots, X_n)$ is a good estimator of θ (this is in fact the MLE).

→ here we know that $\frac{\hat{\theta}_n}{\theta}$ is a pivot, in particular $\mathbb{P}\left(\frac{\hat{\theta}_n}{\theta} < x\right) = x^n$

→ we also know that this simplifies under H_0 to $\mathbb{P}(\hat{\theta}_n < x) = x^n$.

→ moreover, $\mathbb{P}(\hat{\theta}_n \leq \theta) = 1$.

→ we would tend to reject H_0 if $\hat{\theta}_n$ is too far from the value $\theta = 1$, or if $\frac{\hat{\theta}_n}{\theta} \ll 1$ (if $\frac{\hat{\theta}_n}{\theta} \gg 1$, with $\theta = 1$ under H_0, we reject since $\mathbb{P}_{H_0}(\hat{\theta}_n > 1) = 0$)

Hence $\alpha = \mathbb{P}_{H_0}(\text{reject } H_0) = \mathbb{P}_{\theta=1}\left(\frac{\hat{\theta}_n}{\theta} < t_1 \cup \frac{\hat{\theta}_n}{\theta} > t_2\right)$ for $\begin{cases} t_1 < 1 \\ t_2 > 1 \end{cases}$

$$= \mathbb{P}_{\theta=1}\Big(\frac{\hat{\theta}_n}{\theta} < \underset{<1}{t_1}\Big) + \mathbb{P}_{\theta=1}\Big(\frac{\hat{\theta}_n}{\theta} > \underset{>1}{t_2}\Big) \quad \text{with } \mathbb{P}_{\theta=1}\Big(\frac{\hat{\theta}_n}{\theta} > \underset{>1}{t_2}\Big) = 0 \text{ since } X_i \sim \mathcal{U}([0,\theta])$$

$$= \mathbb{P}_{\theta=1}\left(\frac{\hat{\theta}_n}{\theta} < t_1\right) = t_1^n. \text{ We thus have } t_1 = \alpha^{1/n}.$$

And the rejection region thus follows: $R_\alpha(X) = \{X : \hat{\theta}_n < \alpha^{1/n} \text{ or } \hat{\theta}_n > 1\}$.

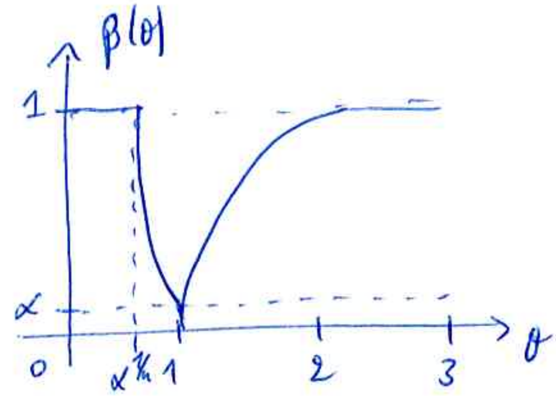
③ - The power function is such that:

$$\beta : \theta \in \Theta_1 = \mathbb{R}^{+*} \setminus \{1\}$$
$$\theta \longmapsto \beta(\theta) = \mathbb{P}_{H_1}(\text{reject } H_0).$$

Then $\beta(\theta) = \mathbb{P}_{H_1}\left(\hat{\theta}_n < \alpha^{1/n} \cup \hat{\theta}_n > 1\right) = \mathbb{P}_{H_1}\left(\hat{\theta}_n < \alpha^{1/n}\right) + \mathbb{P}_{H_1}\left(\hat{\theta}_n > 1\right)$

$$\overset{\theta > 0}{=} \mathbb{P}_{H_1}\left(\frac{\hat{\theta}_n}{\theta} < \frac{\alpha^{1/n}}{\theta}\right) + \mathbb{P}_{H_1}\left(\frac{\hat{\theta}_n}{\theta} > \frac{1}{\theta}\right) = \min\left(\frac{\alpha}{\theta^n}, 1\right) + 1 - \left(\min\left(\frac{1}{\theta}, 1\right)\right)^n$$

Hence
$$\beta(\theta) = \begin{cases} 1 & \text{if } \theta \leq \alpha^{1/n} \\ \frac{\alpha}{\theta^n} & \text{if } \alpha^{1/n} \leq \theta \leq 1 \\ 1 - \frac{1-\alpha}{\theta^n} & \text{if } \theta \geq 1 \end{cases}$$



(4) - Is this test a likelihood-ratio test? (LRT)

First, let us define the likelihood here: $L(\theta; X) = f_{(X_1, \dots, X_n)}((x_1, \dots, x_n); \theta)$

Thus $L(\theta; X) = \prod_{i=1}^{n} f_{X_i}(x_i; \theta)$, with $f_{X_i}(x_i; \theta) = \frac{1}{\theta} \mathbb{1}_{[0;\theta]}(x_i)$

$$= \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(x_i) = \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}_{[0,\theta]}(x_i).$$

$$= \frac{1}{\theta^n} \mathbb{1}_{\{\theta > \max_i (x_i)\}} = \begin{cases} 0 & \text{if } \theta \leq \max_i (x_i) \\ \frac{1}{\theta^n} & \text{if } \theta > \max_i (x_i) \end{cases}$$

This function of θ is decreasing; it takes its maximum in $\hat{\theta}_n = \max_{i=1,\dots,n} (X_i)$.

Our test is based on $\hat{\theta}_n$ rather than $L(\theta; X)$. Indeed, an LRT is defined as the comparison of likelihoods under H_0 and H_1. Our test is therefore not a LRT.

(5) - We have seen that $\hat{\theta}_n = \max(X_1, \dots, X_n)$ is the Maximum Likelihood Estimator (MLE) of θ. It is thus asymptotically gaussian, unbiased, and with a variance that can be approximated by the Fisher information.

Remind that we want to test $H_0 : \theta = 1$ vs $H_1 : \theta \neq 1$.

The Wald test is given by the Wald statistic, saying that under H_0,

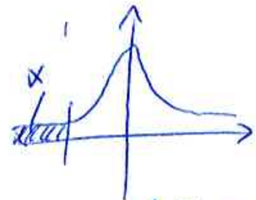
$$\frac{\hat{\theta}_n - 1}{\hat{\sigma}_n} \underset{n \to +\infty}{\overset{H_0}{\sim}} \mathcal{N}(0,1)$$

, with $\hat{\sigma}_n$ an estimator consistent of the standard deviation of $\hat{\theta}_n$.

$\rightarrow \sigma_n^2 = I_n^{-1}(\theta)$ with $I_n(\theta) = \frac{n^2}{\theta^2}$. Hence $\hat{\sigma}_n = \frac{1}{n} \hat{\theta}_n$.

Finally, we have

$$\frac{\hat{\theta}_n - 1}{\frac{1}{n}\hat{\theta}_n} = n\left(1 - \frac{1}{\hat{\theta}_n}\right) \underset{n \to +\infty}{\overset{H_0}{\sim}} \mathcal{N}(0,1),$$



↳ issue if $\hat{\theta}_n$ too small as compared to 1.

which means that $R_\alpha(X) = \left\{ X : n\left(1 - \frac{1}{\hat{\theta}_n(X)}\right) \leq q_\alpha^{\mathcal{N}(0,1)} \right\}$ where $\hat{\theta}_n(X) = \max_{i=1,\dots,n}(X_i)$

Exercice 5 Let X be a random variable related to the weight.

We know that $X_i \sim f$, with $\mathbb{E}[X_i] = \mu$.

① The confidence interval for μ is an asymptotic confidence interval.

Indeed, we use $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ as an estimator of μ. If the X_i's are iid (which is the case here), then $\hat{\mu}_n \underset{n \to +\infty}{\sim} \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$, where σ^2 is the variance of the X_i's (which is unknow here). We know that σ^2 can be consistently estimated by $S_n'^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \hat{\mu}_n)^2$, and we obtain: following the results of the course:

$$CI_{1-\alpha}(\mu) = \left[\hat{\mu}_n - \frac{S_n'}{\sqrt{n}} q_{1-\frac{\alpha}{2}}^{St(n-1)} ; \hat{\mu}_n + \frac{S_n'}{\sqrt{n}} q_{1-\frac{\alpha}{2}}^{St(n-1)} \right]$$

2 _ We set up the test:

$$H_0 : \mu = 500 \quad \text{against} \quad \mu \neq 500.$$

Given the previous question, we can define the rejection region as:

$$R_\alpha(X) = \left\{ X : \underbrace{\frac{|\hat{\mu}_n - 500|}{S'_n/\sqrt{n}}}_{T(X)} \geq q_{1-\frac{\alpha}{2}}^{St(n-1)} \right\}.$$

A.N: $T(x) = \dfrac{|\bar{x}_n - 500|}{s'_n/\sqrt{n}} = \dfrac{|495 - 500|}{5{,}27} \times \sqrt{10} = 3 \geq q_{1-\frac{\alpha}{2}}^{St(n-1)} = 2{,}26.$

$\Rightarrow$ We thus belong to the rejection region $\Rightarrow$ H_0 is rejected!

Exercice 6 We had $X_i^0 \sim \mathcal{N}(\mu_0, \sigma^2)$, with $\mu_0 = 99$ kg.

A new process that is supposed to improve the robustness produces new data x, i.i.d replications of $X_i \sim \mathcal{N}(\mu_1, \sigma_1^2)$.

1 _ We aim to test whether the new process improves the old one, i.e.

$$H_0 : \mu_1 \leq \mu_0 \quad \text{against} \quad H_1 : \mu_1 > \mu_0.$$

Indeed, we would like to reject H_0 to conclude that it is better.

As in the previous exercise, we use $\hat{\mu}_{1n} = \frac{1}{n} \sum_{i=1}^{n} X_i$, a consistent and unbiased estimator of μ_1. We then define the rejection rule as follows:

$$\begin{cases} \text{if } \dfrac{\hat{\mu}_{1n} - \mu_0}{S'_n/\sqrt{n}} > q_{1-\alpha}^{St(n-1)} & \text{then we can reject } H_0 \\ \text{otherwise we cannot reject } H_0. \end{cases}$$

A.N: $\left.\begin{array}{l} \hat{\mu}_{1n} = 102 \\ \mu_0 = 99 \\ s'_n = 2{,}16 \end{array}\right\} \Rightarrow \dfrac{102 - 99}{2{,}16} \sqrt{10} = 4{,}39 \gg 1{,}83 \Rightarrow$ reject H_0.

the new process is therefore better than the old one, with 5% of chance to be mistaken.

② - Is the new process more accurate than the old one?

We have $X_0^i \sim \mathcal{N}(\mu_0, \sigma_0^2 = 1)$, and want to test:
$X_i \sim \mathcal{N}(\mu_1, \sigma_1^2)$
$H_0 : \sigma_1 \geq \sigma_0$ VS $H_1 : \sigma_1^2 < \sigma_0^2$

We know that $S_n'^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \hat{\mu}_{1n})^2$ is a consistent estimator of σ_1^2 - It thus amounts to study the (normalized) "distance" between $S_n'^2$ and σ_0^2. this "distance" is based on the ratio because this is the test statistic whose distribution is known!

To be able to reject H_0 (and conclude H_1), $S_n'^2$ must not be too high as compared to σ_0^2. the rejection zone therefore follows:

$$R_\alpha(X) = \left\{ X : \underbrace{(n-1) \frac{S_n'^2}{\sigma_0^2}}_{T(X)} \leq q_{1-\alpha}^{\chi^2(n-1)} \right\}.$$

A.N: $t(x) = 42$
$q_{1-\alpha}^{\chi^2(n-1)} = 16{,}9$ $\Big\}$ $\Rightarrow t(x) > 16{,}9 = q_{1-\alpha}^{\chi^2_{(n-1)}} \Rightarrow$ we cannot reject H_0.

Exercice 7: Consider an iid sample $(X_i)_{i=1,\dots,n}$ where X_i has density
$$f_\theta(x) = \theta e^{-\theta x} \mathbb{1}_{[0,+\infty[}(x), \text{ with } \theta > 0 \text{ unknown}.$$