

Exercise 1 Could you give the likelihood and/or loglikelihood in the following cases:

Q:

- (1) Poisson distribution, with one single observation $x \Rightarrow X \sim \mathcal{P}(\theta)$
- (2) Exp distribution, with an n -sample: $X = (X_1, \dots, X_n)$, $X_i \sim \text{Exp}(\theta)$
- (3) Cauchy distribution, with a n -sample: $X = (X_1, \dots, X_n)$, $X_i \sim \text{Cauchy}(\theta)$ _{center}
- (4) Geometric distribution, with a n -sample: $X = (X_1, \dots, X_n)$, $X_i \sim \text{Geo}(\theta)$.

A:

$$(1) \cdot L(\theta; x) = f_X(x; \theta) = P(X=x; \theta) = e^{-\theta} \frac{\theta^x}{x!}$$

$$\cdot \ln L(\theta; x) = x \ln(\theta) - \ln(x!) - \theta$$

$$(2) \cdot L(\theta; x) = L(\theta; (x_1, \dots, x_n)) = \int_{(x_1, \dots, x_n)} ((x_1, \dots, x_n); \theta) \stackrel{x_i \perp}{=} \prod_{i=1}^n f_{X_i}(x_i; \theta)$$

$$\stackrel{\text{i.i.d}}{=} \prod_{i=1}^n \theta e^{-\theta x_i} \mathbb{1}_{\{x_i > 0\}} \quad \text{given that } (x_1, \dots, x_n) \text{ is a sequence of positive values.}$$

$$= \theta^n e^{-\theta \sum_{i=1}^n x_i}$$

$$\cdot \ln L(\theta; x) = n \ln(\theta) - \theta \sum_{i=1}^n x_i$$

$$(3) \cdot L(\theta; x) = \prod_{i=1}^n f_{X_i}(x_i; \theta) = \prod_{i=1}^n \frac{1}{\pi(1+(x_i-\theta)^2)} \mathbb{1}_{\mathbb{R}}(x_i)$$

$$= \pi^{-n} \prod_{i=1}^n \frac{1}{1+(x_i-\theta)^2}$$

$$\cdot \ln L(\theta; x) = -n \ln(\pi) + \sum_{i=1}^n \ln\left(\frac{1}{1+(x_i-\theta)^2}\right) = -n \ln(\pi) - \sum_{i=1}^n \ln(1+(x_i-\theta)^2)$$

$$(4) \cdot L(\theta; x) = \prod_{i=1}^n P(X_i = x_i; \theta) = \prod_{i=1}^n (1-\theta)^{x_i-1} \theta = \theta^n \prod_{i=1}^n \frac{(1-\theta)^{x_i}}{(1-\theta)}$$

$$= \left(\frac{\theta}{1-\theta}\right)^n (1-\theta)^{\sum_{i=1}^n x_i}$$

$$\cdot \ln L(\theta; x) = n \ln(\theta) - n \ln(1-\theta) + \left(\sum_{i=1}^n x_i\right) \ln(1-\theta)$$

$$= n \ln(\theta) + \left(\sum_{i=1}^n x_i - n\right) \ln(1-\theta).$$

Exercice 2 : on the computer...

Exercice 3 Compute the maximum likelihood estimator from a n-sample
Q: in the following cases:

(1) $f(x; \theta) = \theta x^{\theta-1} \mathbb{1}_{\{0 < x < 1\}}$, with $\theta > 0$.

(2) $f(x; \theta) = \theta^2 x e^{-\theta x} \mathbb{1}_{\{x > 0\}}$, with $\theta > 0$.

(3) $f(x; \theta) = (\theta+1) x^{-\theta-2} \mathbb{1}_{\{x > 1\}}$, with $\theta > 0$.

A: (1) $\underset{\text{log-likelihood}}{L(\theta; x)} = \sum_{i=1}^n \ln(f_{X_i}(x_i; \theta)) = \sum_{i=1}^n \ln(\theta x_i^{\theta-1}) = \sum_{i=1}^n [\ln(\theta) + (\theta-1) \ln(x_i)]$

$$= n \ln(\theta) + (\theta-1) \sum_{i=1}^n \ln(x_i)$$

$\hat{\theta}$ is such that $\frac{\partial L(\theta; x)}{\partial \theta} \Big|_{\theta=\hat{\theta}} = 0 \Leftrightarrow \frac{n}{\hat{\theta}} + \sum_{i=1}^n \ln(x_i) = 0 \Rightarrow \hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln(x_i)}$

(2) $L(\theta; x) = \sum_{i=1}^n \ln(\theta^2 x_i e^{-\theta x_i}) = \sum_{i=1}^n [2 \ln(\theta) + \ln(x_i) - \theta x_i]$

$$= 2n \ln(\theta) + \sum_{i=1}^n \ln(x_i) - \theta \sum_{i=1}^n x_i$$

Then $\frac{2n}{\hat{\theta}} - \sum_{i=1}^n x_i = 0 \Leftrightarrow \hat{\theta} = \frac{2n}{\sum_{i=1}^n x_i}$

$$(3) \ell(\theta; x) = \sum_{i=1}^n \ln \left((\theta+1) x_i^{-(\theta+2)} \right) = \sum_{i=1}^n \left[\ln(\theta+1) - (\theta+2) \ln(x_i) \right] \quad (2)$$

$$= n \ln(\theta+1) - (\theta+2) \sum_{i=1}^n \ln(x_i)$$

Then $\frac{n}{\hat{\theta}+1} - \sum_{i=1}^n \ln(x_i) = 0 \Leftrightarrow \frac{n}{\hat{\theta}+1} = \sum_{i=1}^n \ln(x_i) \Leftrightarrow \hat{\theta}+1 = \frac{n}{\sum_{i=1}^n \ln(x_i)}$

Hence $\hat{\theta} = \frac{n}{\sum_{i=1}^n \ln(x_i)} - 1$

Exercise 4 Compute the Fisher information when dealing with a
 Q: n -sample $X = (X_1, \dots, X_n)$, where (1) $X_i \sim \mathcal{B}(\pi)$
 (2) $X_i \sim \mathcal{N}(\mu, \sigma^2)$

A: (1) We have a n -sample, i.e. $(X_i)_{1 \leq i \leq n}$ are iid realizations, and because they are independent r.v. we know that $I_n(\pi) = n I_1(\pi)$.

Moreover, by definition, $I_n(\pi) = \text{Var} \left(\frac{\partial}{\partial \pi} \ln L(\pi; (X_1, \dots, X_n)) \right)$

Sometimes, it is simpler to use:

$$I_n(\pi) = - \mathbb{E} \left[\frac{\partial^2}{\partial \pi^2} \ln L(\pi; (X_1, \dots, X_n)) \right]$$

Here, we have:

$$\ell(\pi; x_1) = \ln(L(\pi; x_1)) = \ln(\pi^{x_1} (1-\pi)^{1-x_1}) = x_1 \ln(\pi) + (1-x_1) \ln(1-\pi)$$

therefore $I_n(\pi) = n I_1(\pi) = n \text{Var} \left(\frac{\partial}{\partial \pi} \ell(\pi; x_1) \right) = n \text{Var} \left(\frac{x_1}{\pi} - \frac{1-x_1}{1-\pi} \right)$

$$= n \left[\frac{1}{\pi^2} \text{Var}(X_1) + \frac{1}{(1-\pi)^2} \text{Var}(X_1) \right] = n \left(\frac{\pi(1-\pi)}{\pi^2} + \frac{\pi(1-\pi)}{(1-\pi)^2} \right)$$

$$= n \left(\frac{1-\pi}{\pi} + \frac{\pi}{1-\pi} \right) = n \left(\frac{(1-\pi)^2 + \pi^2}{\pi(1-\pi)} \right) = n \left(\frac{1-2\pi+\pi^2+\pi^2}{\pi(1-\pi)} \right) = \frac{n}{\pi(1-\pi)}$$

$$(2): X_i \sim \mathcal{N}(\mu, \sigma^2).$$

Here θ is a two-dimensional vector of parameters. The definition thus has to be adapted, considering partial derivatives. Taking

$$\mathcal{I}_1(\theta) = \mathcal{I}_1(\mu, \sigma^2) = -E \left[\begin{pmatrix} \frac{\partial^2}{\partial \mu^2} \ln L(\theta; X) & \frac{\partial^2}{\partial \mu \partial \sigma^2} \ln L(\theta; X) \\ \frac{\partial^2}{\partial \sigma^2 \partial \mu} \ln L(\theta; X) & \frac{\partial^2}{(\partial \sigma^2)^2} \ln L(\theta; X) \end{pmatrix} \right]$$

$$\bullet \ell(\theta; X_1) = \ln L(\mu, \sigma^2; X_1)$$

$$= \ln f(X_1; (\mu, \sigma^2))$$

$$= \ln \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_1 - \mu)^2}{2\sigma^2}} \right) = -\frac{1}{2} \ln(2\pi) - \frac{1}{2} \ln(\sigma^2) - \frac{(X_1 - \mu)^2}{2\sigma^2}$$

$$\bullet \frac{\partial}{\partial \mu} \ell(\theta; X_1) = \cancel{(-1)} \mathcal{E}(X_1 - \mu) \frac{1}{\sigma^2} = \frac{X_1 - \mu}{\sigma^2}$$

$$\bullet \frac{\partial^2}{\partial \mu \partial \sigma^2} \ell(\theta; X_1) = (X_1 - \mu) \times (\sigma^2)^{-2} = -\frac{X_1 - \mu}{(\sigma^2)^2} = \frac{\mu - X_1}{\sigma^4} \Rightarrow -E \left[\frac{\mu - X_1}{\sigma^4} \right] = 0$$

$$\bullet \frac{\partial^2}{\partial \mu^2} \ell(\theta; X_1) = -\frac{1}{\sigma^2} \Rightarrow -E \left[-\frac{1}{\sigma^2} \right] = \frac{1}{\sigma^2}$$

$$\bullet \frac{\partial}{\partial \sigma^2} \ell(\theta; X_1) = -\frac{1}{2\sigma^2} - \frac{(X_1 - \mu)^2}{2} \times (-1)(\sigma^2)^{-2} = -\frac{1}{2\sigma^2} + \frac{(X_1 - \mu)^2}{2\sigma^4}$$

$$\bullet \frac{\partial^2}{(\partial \sigma^2)^2} \ell(\theta; X_1) = -\frac{1}{2} (-1)(\sigma^2)^{-2} + \frac{(X_1 - \mu)^2}{2} (-2)(\sigma^2)^{-3}$$

$$= \frac{1}{2\sigma^4} - \frac{(X_1 - \mu)^2}{\sigma^6} \Rightarrow -E \left[\frac{1}{2\sigma^4} - \frac{(X_1 - \mu)^2}{\sigma^6} \right] = -\frac{1}{2\sigma^4} + \frac{1}{\sigma^6} \frac{E[(X_1 - \mu)^2]}{\text{Var}(X_1)}$$

$$= -\frac{1}{2\sigma^4} + \frac{1}{\sigma^6} \sigma^2 = -\frac{1}{2\sigma^4} + \frac{1}{\sigma^4} = \frac{-1+2}{2\sigma^4} = \frac{1}{2\sigma^4}$$

$$\text{We then get the result: } \mathcal{I}_n(\mu, \sigma^2) = \begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$$

Exercise 5 let $x = (x_1, \dots, x_n)$ a n -sample of iid v.v. such that (3)

$X_i \sim \mathcal{B}(\pi)$. Show that $S = X_1 + X_2 + \dots + X_n$ is an exhaustive statistic. Is it also minimal?

A: $X_i \sim \mathcal{B}(\pi)$, $X_i \in \{0, 1\}$, $P(X_i = x_i) = \pi^{x_i} (1-\pi)^{1-x_i}$

$$L(\pi; x) = \prod_{i=1}^n P(X_i = x_i) = \prod_{i=1}^n \pi^{x_i} (1-\pi)^{1-x_i} = \pi^{\sum_{i=1}^n x_i} (1-\pi)^{n - \sum_{i=1}^n x_i}$$

We can conclude from the factorization theorem by Fisher-Neyman; since

there exist two measurable functions g and h such that: $\forall x \in \mathcal{X}, \forall \pi \in \Pi$,

$$L(\pi; x) = \underbrace{\left(\frac{\pi}{1-\pi} \right)^{\sum x_i} (1-\pi)^n}_{g(\sum x_i; \pi)} \times \underbrace{1}_{h(x)} \Rightarrow t(x) = \sum_{i=1}^n x_i \text{ is an exhaustive statistic for } \pi.$$

this statistic is also minimal, Indeed, we need at least this information to be able to compute the likelihood.

• Same question with $X_i \sim \mathcal{N}(\theta, 1)$:

$$\begin{aligned} L(\theta; x) &= \prod_{i=1}^n f_{X_i}(x_i; \theta) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i - \theta)^2}{2}} = \left(\frac{1}{\sqrt{2\pi}} \right)^n e^{-\sum_{i=1}^n \frac{(x_i - \theta)^2}{2}} \\ &= (2\pi)^{-\frac{n}{2}} e^{-\sum_{i=1}^n \frac{x_i^2 - 2\theta x_i + \theta^2}{2}} = (2\pi)^{-\frac{n}{2}} \underbrace{e^{-\frac{1}{2} \sum_{i=1}^n x_i^2}}_{h(x)} \underbrace{e^{2\theta \sum_{i=1}^n x_i} e^{-\frac{n}{2} \theta^2}}_{g(\sum x_i, \theta)} \end{aligned}$$

Therefore, $\sum_{i=1}^n X_i = T(x)$ is a sufficient (exhaustive) statistic for the estimation of θ . It is also minimal.

Exercise 6 let $X = (X_1, \dots, X_n)$ a n -sized sample, where $X_i \sim \mathcal{U}(0; \theta)$.

Denote by $\begin{cases} \hat{\theta}_n = \max(X_1, \dots, X_n) \\ \tilde{\theta}_n = 2\bar{X}_n = \frac{2}{n} \sum_{i=1}^n X_i \end{cases}$

- 1) Show that $\tilde{\theta}_n$ is an unbiased and consistent estimator of θ , and that $V_n(\tilde{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \frac{\theta^2}{3})$.

A: • $E[\tilde{\theta}_n] = E[2\bar{X}_n] = 2E[\bar{X}_n] = 2E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{2}{n} \sum_{i=1}^n E[X_i]$
 $= \frac{2n}{n} E[X_i] = 2E[X_i] = 2\left(\frac{b-a}{2}\right)$ with $\begin{cases} a=0 \\ b=\theta \end{cases}$
 $= 2 \frac{\theta}{2} = \theta \Rightarrow \text{unbiased.}$

- $\tilde{\theta}_n$ is unbiased: we now check the variance to study the quadratic risk:

$$\begin{aligned} \text{Var}(\tilde{\theta}_n) &= \text{Var}(2\bar{X}_n) = 4 \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{4}{n^2} \text{Var}\left(\sum X_i\right) = \frac{4}{n} \text{Var}(X_i) \\ &= \frac{4}{n} \frac{(b-a)^2}{12} = \frac{4}{12n} \theta^2 \xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

then $R(\tilde{\theta}_n) = E[(\tilde{\theta}_n - \theta)^2] = \text{Var}(\tilde{\theta}_n) \xrightarrow[n \rightarrow \infty]{} 0 \Rightarrow \tilde{\theta}_n \xrightarrow[n \rightarrow \infty]{\text{MR}} \theta$

And therefore $\tilde{\theta}_n \xrightarrow[n \rightarrow \infty]{\text{IP}} \theta$. $\tilde{\theta}_n$ is a consistent estimator of θ .

- X_i are iid, with finite mean m_i and variance σ_i^2 from the CLT,

$$\sum_{i=1}^n X_i \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(nm_i, n\sigma_i^2) \Rightarrow \sum_{i=1}^n X_i \sim \mathcal{N}\left(n \frac{\theta}{2}, n \frac{\theta^2}{12}\right)$$

$$\Rightarrow \underbrace{\frac{2}{n} \sum X_i}_{\tilde{\theta}_n} \sim \mathcal{N}\left(\theta, \frac{\theta^2}{3n}\right) \Rightarrow V_n(\tilde{\theta}_n - \theta) \sim \mathcal{N}\left(0, \frac{\theta^2}{3}\right).$$

(2) Build an asymptotic confidence interval on θ , based on $\tilde{\theta}_n$, with security coefficient $1-\alpha$. ④

We thus look for bounds L and U such that:

$$P(L \leq \theta \leq U) = 1-\alpha$$

$$P(\tilde{\theta}_n - U \leq \tilde{\theta}_n - \theta \leq \tilde{\theta}_n - L) = 1-\alpha$$

$$P(\sqrt{n}(\tilde{\theta}_n - U) \leq \sqrt{n}(\tilde{\theta}_n - \theta) \leq \sqrt{n}(\tilde{\theta}_n - L)) = 1-\alpha$$

$$P\left(\frac{\sqrt{n}(\tilde{\theta}_n - U)}{\sqrt{\frac{\sigma^2}{3}}} \leq \underbrace{\frac{\sqrt{n}(\tilde{\theta}_n - \theta)}{\sqrt{\frac{\sigma^2}{3}}}}_{Z \underset{n \rightarrow \infty}{\sim} \mathcal{N}(0,1)} \leq \frac{\sqrt{n}(\tilde{\theta}_n - L)}{\sqrt{\frac{\sigma^2}{3}}}\right) = 1-\alpha$$

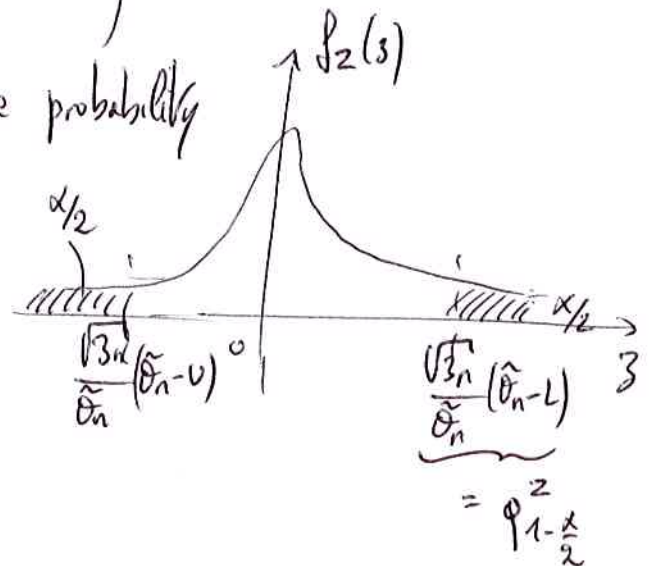
$$P\left(\frac{\sqrt{3n}}{\sigma}(\tilde{\theta}_n - U) \leq Z \leq \frac{\sqrt{3n}}{\sigma}(\tilde{\theta}_n - L)\right) = 1-\alpha$$

We know that $\tilde{\theta}_n \xrightarrow[n \rightarrow \infty]{P} \theta$, thus we use the Slutsky Theorem:

$$F_2\left(\frac{\sqrt{3n}}{\tilde{\theta}_n}(\tilde{\theta}_n - L)\right) - F_2\left(\frac{\sqrt{3n}}{\tilde{\theta}_n}(\tilde{\theta}_n - U)\right) = 1-\alpha$$

Z has a symmetric density \Rightarrow we can share probability α between two different zones with same probability $\frac{\alpha}{2} \Rightarrow L = -U$

We thus have



$$\underbrace{q_{0,975}^Z}_{=1,96} = q_{1-\frac{\alpha}{2}}^Z = \frac{\sqrt{3n}}{\tilde{\theta}_n}(\tilde{\theta}_n - L)$$

therefore $\tilde{\theta}_n \varphi_{1-\frac{\alpha}{2}}^z = \sqrt{3n} (\tilde{\theta}_n - L) \Leftrightarrow \frac{\tilde{\theta}_n}{\sqrt{3n}} \varphi_{1-\frac{\alpha}{2}}^z = \tilde{\theta}_n - L$

Finally,
$$\begin{cases} L = \tilde{\theta}_n - \frac{\tilde{\theta}_n}{\sqrt{3n}} \varphi_{1-\frac{\alpha}{2}}^z \\ U = \tilde{\theta}_n + \frac{\tilde{\theta}_n}{\sqrt{3n}} \varphi_{1-\frac{\alpha}{2}}^z \end{cases} \Rightarrow IC_{1-\alpha}(\theta) = \left[\tilde{\theta}_n \pm \frac{\tilde{\theta}_n}{\sqrt{3n}} \varphi_{1-\frac{\alpha}{2}}^z \right]$$

(3) Show that $\frac{\hat{\theta}_n}{\theta}$ is a pivotal function to estimate θ : (remind that a pivotal function has a distribution which does not depend on θ).

A: We have $\hat{\theta}_n = \max(X_1, \dots, X_n)$, where $X_i \stackrel{iid}{\sim} \mathcal{U}([0, \theta])$.

This means that $IP\left(\frac{\hat{\theta}_n}{\theta} \leq x\right) = IP\left(\frac{\max(X_1, \dots, X_n)}{\theta} \leq x\right) = IP\left(\frac{X_1}{\theta} \leq x, \dots, \frac{X_n}{\theta} \leq x\right)$

$$= IP\left(\frac{X_1}{\theta} \leq x, \dots, \frac{X_n}{\theta} \leq x\right), \text{ where } \frac{X_i}{\theta} \sim \mathcal{U}([0, 1])$$

Hence
$$IP\left(\frac{\hat{\theta}_n}{\theta} \leq x\right) \stackrel{II}{=} IP\left(\frac{X_1}{\theta} \leq x\right) IP\left(\frac{X_2}{\theta} \leq x\right) \dots IP\left(\frac{X_n}{\theta} \leq x\right)$$

$$= IP(U_1 \leq x) \dots IP(U_n \leq x) = x^n \text{ if } 0 \leq x \leq 1.$$

Finally,
$$F_{\frac{\hat{\theta}_n}{\theta}}(x) = IP\left(\frac{\hat{\theta}_n}{\theta} \leq x\right) = \begin{cases} 0 & \text{if } x < 0 \\ x^n & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases} \Rightarrow \text{this distribution does not depend on } \theta.$$

(4) Build a (non-asymptotic) confidence interval of θ , with security coefficient $1-\alpha$, based on $\hat{\theta}_n$.

We look for bounds U and L such that:

$$IP(L \leq \theta \leq U) = 1-\alpha.$$

We know that $P(\theta > 0) = 1$. (5)

We thus get $P(0 \leq \theta \leq U) = 1 - \alpha \Leftrightarrow P(\theta \leq U) - \overbrace{P(\theta \leq 0)}^{=0} = 1 - \alpha$

And then $P\left(\frac{1}{\theta} \geq \frac{1}{U}\right) = 1 - \alpha \Leftrightarrow 1 - P\left(\frac{\hat{\theta}_n}{\theta} \leq \frac{\hat{\theta}_n}{U}\right) = 1 - \alpha$

Therefore $\left(\frac{\hat{\theta}_n}{U}\right)^n = \alpha \Leftrightarrow \frac{\hat{\theta}_n}{U} = \alpha^{1/n} \Leftrightarrow U = \frac{\hat{\theta}_n}{\sqrt[n]{\alpha}}$

We thus have $IC(\theta) = \left[0, \frac{\hat{\theta}_n}{\alpha^{1/n}}\right]$

(5) Compare the two obtained intervals:

$$IC_1 = \left[\hat{\theta}_n - \frac{\tilde{\theta}_n}{\sqrt{3n}} z_{1-\frac{\alpha}{2}}, \hat{\theta}_n + \frac{\tilde{\theta}_n}{\sqrt{3n}} z_{1-\frac{\alpha}{2}} \right]$$

$$IC_2 = \left[0, \frac{\hat{\theta}_n}{\alpha^{1/n}} \right]$$

Exercice 7 Let $\theta \in]0, 1[$, and X be a discrete random variable with mass function: $P_\theta(X=k) = (k+1)(1-\theta)^2 \theta^k$, $k \in \mathbb{N}$.

We have $E[X] = \frac{2\theta}{1-\theta}$ and $Var(X) = \frac{2\theta}{(1-\theta)^2}$.

The goal is to estimate θ from the sample of iid r.v. $X = (X_1, \dots, X_n)$.

- ① - Give $\hat{\theta}_n$ the moment estimator of θ .
- ② - Is $\tilde{\theta}_n$ the maximum likelihood estimator well defined?
- ③ - Study the consistency of $\hat{\theta}_n$, and give its limiting distribution.

A: ① - By the SLLN, we have $\bar{X}_n \xrightarrow[n \rightarrow \infty]{P.s.} E[X]$.

Here we have $E[X] = \frac{2\theta}{1-\theta} \Leftrightarrow 2\theta = (1-\theta)E[X]$

$\Leftrightarrow \theta(2+E[X]) = E[X] \Leftrightarrow \theta = \frac{E[X]}{2+E[X]}$.

We thus get $\hat{\theta}_n = \frac{\bar{X}_n}{2+\bar{X}_n}$.

② - $l(\theta; x) = \ln L(\theta; x) = \sum_{i=1}^n \ln p_{\theta}(X_i) = \sum_{i=1}^n \ln((X_i+1)(1-\theta)^2 \theta^{X_i})$
 $= \ln \ln(1-\theta) + n \bar{X}_n \ln(\theta) + \sum_{i=1}^n \ln(X_i+1)$

$\Rightarrow \frac{\partial l(\theta; x)}{\partial \theta} = -\frac{2n}{1-\theta} + n \frac{\bar{X}_n}{\theta}$.

is this function is positive? $-\frac{2n}{1-\theta} + n \frac{\bar{X}_n}{\theta} \geq 0 \Leftrightarrow n \frac{\bar{X}_n}{\theta} \geq \frac{2n}{1-\theta}$

$\Leftrightarrow \frac{1-\theta}{\theta} \geq \frac{2}{\bar{X}_n} \Leftrightarrow \frac{1}{\theta} - 1 \geq \frac{2}{\bar{X}_n} \Leftrightarrow \frac{1}{\theta} \geq 1 + \frac{2}{\bar{X}_n}$

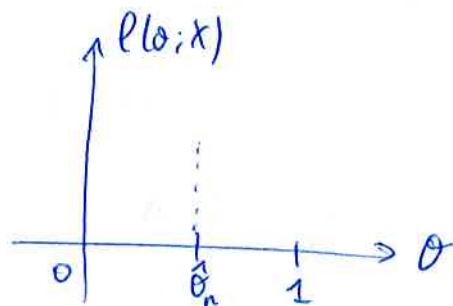
$\Leftrightarrow \theta \leq \left(1 + \frac{2}{\bar{X}_n}\right)^{-1} = \left(\frac{n\bar{X}_n + 2n}{n\bar{X}_n}\right)^{-1} = \frac{\bar{X}_n}{2+\bar{X}_n} = \hat{\theta}_n$.
Method
(MME) Estimator
Moment

Hence $l'(\theta; x) \geq 0 \Leftrightarrow \theta \leq \hat{\theta}_n = \text{MME}$

• If $\hat{\theta}_n > 0$ (i.e. $\bar{X}_n > 0$), then the maximum is located in $\hat{\theta}_n$ on $]0, 1[$.

Then $\tilde{\theta}_n = \hat{\theta}_n$.

• If $\hat{\theta}_n = 0$ (i.e. $\bar{X}_n = 0$), then the likelihood is ^{strictly} decreasing on $]0, 1[$ and the MLE ~~is not~~ does not exist.



③ = We know that $\bar{X}_n \xrightarrow[n \rightarrow \infty]{p.s.} E[X_i] = E[X]$. ⑥

By the continuity theorem, we thus have $\hat{\theta}_n \xrightarrow[n \rightarrow \infty]{p.s.} \theta$.

Then $\hat{\theta}_n$ is consistent. Moreover, provided that $\begin{pmatrix} \text{Var}(X_i) < +\infty \\ E[X_i] < +\infty \\ X_i \text{ iid.} \end{pmatrix}$, we have

from CLT: $\sum_{i=1}^n X_i \underset{n \rightarrow \infty}{\sim} \mathcal{N}\left(\sum_{i=1}^n E[X_i], \sum_{i=1}^n \text{Var}(X_i)\right)$.

$$\bar{X}_n \sim \mathcal{N}\left(E[X_i], \frac{\text{Var}(X_i)}{n}\right) \Leftrightarrow \bar{X}_n \underset{n \rightarrow \infty}{\sim} \mathcal{N}\left(\frac{2\theta}{1-\theta}, \frac{2\theta}{n(1-\theta)^2}\right)$$

$$\sqrt{n}\left(\bar{X}_n - \frac{2\theta}{1-\theta}\right) \underset{n \rightarrow \infty}{\sim} \mathcal{N}\left(0, \frac{2\theta}{(1-\theta)^2}\right), \text{ then we use the Delta method}$$

$$\text{with function } g(x) = \frac{x}{2+x} = \frac{2+x-2}{2+x} = 1 - \frac{2}{2+x}, \text{ at point } \theta' = \frac{2\theta}{1-\theta}.$$

This function has a derivative in $x = \frac{2\theta}{1-\theta}$, with derivative given by:

$$g'(x) = \frac{2}{(2+x)^2}, \text{ thus } g'\left(\frac{2\theta}{1-\theta}\right) = \frac{2}{\left(2 + \frac{2\theta}{1-\theta}\right)^2} = \frac{(1-\theta)^2}{2}$$

$$\text{Finally, we get } \sqrt{n}(\hat{\theta}_n - \theta) \underset{n \rightarrow \infty}{\sim} \mathcal{N}\left(0, \frac{2\theta}{(1-\theta)^2} \times \left(\frac{(1-\theta)^2}{2}\right)^2 = \mathcal{N}\left(0, \frac{\theta(1-\theta)^2}{2}\right).$$

Exercise 8 Let $X = (X_1, \dots, X_{25})$ an iid. sample of random variables with

$X_i \sim \mathcal{N}(\mu, \sigma^2 = 81)$. We know that $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ gave

a realization $\bar{x}_n = 81,2$.

Find the confidence interval with confidence probability 95% for the mean μ .

This is the case of Gaussian random variable with known variance.

Hence, $\bar{X}_n \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}) \Leftrightarrow \sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

(Remember this is an exact confidence interval, not an asymptotic one!).

Therefore $\sqrt{n} \frac{\bar{X}_n - \mu}{\sigma}$ is a pivot. (a function of the observations X_i and the parameter of interest μ , whose distribution is independent from μ).

Thus, we look for bounds Z_1 and Z_2 such that

$$P(Z_1 \leq \mu \leq Z_2) = 1 - \alpha.$$

But we know that $U = \sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} \sim \mathcal{N}(0, 1)$, thus

$$P(|U| \leq \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}) = 1 - \alpha$$

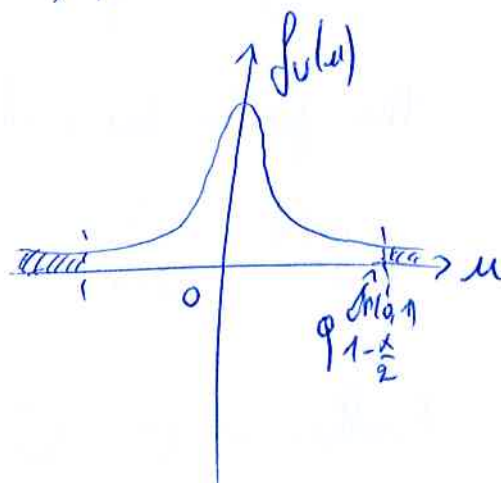
$$\Rightarrow P\left(\sqrt{n} \left| \frac{\bar{X}_n - \mu}{\sigma} \right| \leq \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\right) = 1 - \alpha$$

$$\Rightarrow P\left(|\bar{X}_n - \mu| \leq \frac{\sigma}{\sqrt{n}} \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\underbrace{\bar{X}_n - \frac{\sigma}{\sqrt{n}} \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}}_{Z_1} \leq \mu \leq \underbrace{\bar{X}_n + \frac{\sigma}{\sqrt{n}} \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}}_{Z_2}\right) = 1 - \alpha$$

$$\Rightarrow IC_{1-\alpha}(\mu) = \left[\bar{X}_n - \frac{\sigma}{\sqrt{n}} \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} ; \bar{X}_n + \frac{\sigma}{\sqrt{n}} \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} \right].$$

A.N.: $\left(\varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} = \varphi_{97.5\%}^{\mathcal{N}(0,1)} = 1.96 \right)$
 $n=25, \sigma=9, \bar{x}_n=81.2 \Rightarrow IC_{1-\alpha}(\mu) \cong \left[81.2 - \frac{9}{5} \times 2, 81.2 + \frac{9}{5} \times 2 \right]$



Exercice 9

Let $X = (X_1, \dots, X_n)$ a n-sample of i.i.d random variables, where $X_i \sim \mathcal{N}(\mu, \sigma^2 = 16)$. Find the minimal number of observations such that $[\bar{X}_n - 1, \bar{X}_n + 1]$ would be the 90% confidence interval of μ .

A: We are exactly in the same framework as in Exercice 7. Thus,

$$\left[\bar{X}_n - \frac{\sigma}{\sqrt{n}} \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} ; \bar{X}_n + \frac{\sigma}{\sqrt{n}} \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} \right] = [\bar{X}_n - 1 ; \bar{X}_n + 1].$$

Therefore, we can identify the bounds; which yields:

$$\frac{\sigma}{\sqrt{n}} \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} = 1 \Leftrightarrow \sqrt{n} = \sigma \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} \Leftrightarrow n = \left(\sigma \varphi_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} \right)^2$$

A.N: $\left. \begin{array}{l} \alpha = 10\% \\ \sigma = 4 \\ \varphi_{95\%}^{\mathcal{N}(0,1)} = 1,64 \end{array} \right\} \Rightarrow n = (4 \times 1,64)^2 = 6,56^2 \approx 43, \dots$

We thus require $n \geq 44$.