Exercice 1 Q: What is the density of (X,Y), whose joint cumulative distribution function is given by:

$$F(x,y) = \begin{cases} 0 & \text{if } x<0 \\ (1-e^{-x})\left(\frac{1}{2}+\frac{1}{\pi}\arctan(y)\right) & \text{otherwise} \end{cases}$$

A: The joint density is given by:

$$f(x,y) = \frac{\partial F(x,y)}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\left(\frac{1}{2}+\frac{1}{\pi}\arctan(y)\right)e^{-x}\right) = \frac{\partial}{\partial y}\left(\frac{e^{-x}}{2}+\frac{e^{-x}}{\pi}\arctan y\right)$$

$$= \frac{1}{(1+y^2)\pi}e^{-x} \Rightarrow f(x,y) = \frac{1}{\pi(1+y^2)}e^{-x}\mathbb{1}_{\{x\geq 0\}}$$

- $f(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$
- $\iint_{\mathbb{R}^2} f(x,y) = 1.$

Exercice 2 For the following functions F, which of them correspond to cdf of (X,Y)?

- $F(x,y) = \begin{cases} 1-e^{-x-y} & \text{if } (x,y)\in\mathbb{R}^{+2} \\ 0 & \text{otherwise} \end{cases} \Rightarrow f(x,y) = \frac{\partial}{\partial x\partial y}F(x,y) = \frac{\partial}{\partial y\partial x}F(x,y)$

$$= \begin{cases} -e^{-x-y} & \text{if } (x,y)\in\mathbb{R}^{+2} \\ 0 & \text{otherwise} \end{cases}$$

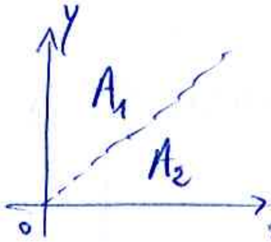
Here, $f(x,y) \leq 0 \Rightarrow$ this is not a density!

- $F(x,y) = \begin{cases} 1-e^{-x}-xe^{-y} & \text{if } 0\leq x\leq y \\ 1-e^{-y}-ye^{-x} & \text{if } 0\leq y\leq x \\ 0 & \text{otherwise} \end{cases}$

→ If $0 \le x \le y$: $f(x,y) = \frac{\partial F(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left(e^{-x} - e^{-y}\right) = e^{-y}$ (in area A_1)

→ If $0 \le y \le x$: $f(x,y) = \frac{\partial F(x,y)}{\partial y \partial x} = e^{-x}$ by symmetry. (in area A_2)

→ otherwise $f(x,y) = 0$



- We have that for all $(x,y) \in \mathbb{R}^2$, $f(x,y) \ge 0$.
- Study: $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(e^{-y}\mathbb{1}_{\{(x,y)\in A_1\}} + e^{-x}\mathbb{1}_{\{(x,y)\in A_2\}}\right)dx\,dy$

$$= \iint_{\overline{A_1 \cup A_2}} 0\,dx\,dy + \iint_{A_1} e^{-y}\,dx\,dy + \iint_{A_2} e^{-x}\,dx\,dy$$

$$= \int_0^{+\infty}\left(\int_0^y e^{-y}\,dx\right)dy + \int_0^{+\infty}\left(\int_0^x e^{-x}\,dy\right)dx = \int_0^{+\infty}\left[xe^{-y}\right]_0^y dy + \int_0^{+\infty}\left[ye^{-x}\right]_0^x dx$$

$$= \int_0^{+\infty} \underbrace{y}_{u}\,\underbrace{e^{-y}}_{v'}\,dy + \int_0^{+\infty} xe^{-x}\,dx \overset{IPP}{=} \left(\left[-ye^{-y}\right]_0^{+\infty} + \int_0^{+\infty} e^{-y}\,dy\right) + \left(\left[-xe^{-x}\right]_0^{+\infty} + \int_0^{+\infty} e^{-x}\,dx\right)$$

$$= \left((0+0) + \left[-e^{-y}\right]_0^{+\infty}\right) + \left((0+0) + \left[-e^{-x}\right]_0^{+\infty}\right) = 1 + 1 = 2.$$

⇒ f is not a density function, and thus F is not a cdf.

Exercise 3 Let (X,Y,Z) be a real-valued random vector with (2)

density function given by:

$$f_{(X,Y,Z)}(x,y,z) = \begin{cases} (y-x)^2 e^{-(1+z)(y-x)} & \text{if } 0 \le x \le 1,\ y \geq x \text{ and } z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Consider the new vector (U,V,W) such that: $\begin{cases} U = X \\ V = Y-X \\ W = Z(Y-X) \end{cases}$.

What is the distribution of (U,V,W)?

A: Reminder: in the 1-dimensional case; consider a monotonous increasing transformation ϕ such that $Y = \phi(X)$, where we know the distribution of X. Then, $F_Y(x) = \mathbb{P}(Y \le x) = \mathbb{P}(\phi(X) \le x) = \mathbb{P}(X \le \phi^{-1}(x))$

$= F_X(\phi^{-1}(x))$

Hence, $f_Y(x) = \frac{d}{dx} F_Y(x) = (\phi^{-1}(x))' f_X(\phi^{-1}(x))$ (I)

It means that we can recover the distribution of Y knowing the distribution of X. A simple example is the lognormal distribution:

Here, $(U,V,W) = \phi((X,Y,Z)) \Rightarrow (X,Y,Z) = \phi^{-1}((U,V,W))$ where

$$\begin{cases} X = U \\ Y = V + X = V + U \\ Z = \frac{W}{Y-X} = \frac{W}{V} \end{cases}$$

$\Rightarrow (\phi^{-1})'$ is called the Jacobian since we are in the multivariate case:

$$J = \begin{pmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} & \frac{\partial X}{\partial W} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} & \frac{\partial Y}{\partial W} \\ \frac{\partial Z}{\partial U} & \frac{\partial Z}{\partial V} & \frac{\partial Z}{\partial W} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -\frac{W}{V^2} & \frac{1}{V} \end{pmatrix}$$

Therefore, $|\det J| = \left|\frac{1}{v}\right| = \frac{1}{|v|} = \frac{1}{v}$ since $v \geq 0$.

Indeed, $\begin{cases} 0 \leq x \leq 1 \Rightarrow 0 \leq u \leq 1 & \text{since } X = U \\ 0 \leq x \leq y \Rightarrow 0 \leq y - x \rightarrow v \geq 0 & \text{since } V = Y - X \\ z \geq 0 \Rightarrow z(y-x) \geq 0 \Rightarrow w \geq 0 \end{cases}$

Using formula (I) generalized to the multivariate case, we obtain:

$$f_{(U,V,W)}(u,v,w) = \underbrace{\frac{1}{v}}_{=|\det J|} f_{(X,Y,Z)}\Big(\underbrace{u}_{x}, \underbrace{u+v}_{y}, \underbrace{\frac{w}{v}}_{z}\Big)$$

$$= \frac{1}{v} v^2 e^{-\left(1+\frac{w}{v}\right)v} \mathbb{1}_{\{0 \leq u \leq 1,\, v \geq 0,\, w \geq 0\}}(u,v,w)$$

$$= v e^{-w-v} \mathbb{1}_{\{0 \leq u \leq 1,\, v \geq 0,\, w \geq 0\}}(u,v,w)$$

$$= \underbrace{v e^{-v} \mathbb{1}_{\{v \geq 0\}}(v)}_{g_1(v)} \underbrace{e^{-w} \mathbb{1}_{\{w \geq 0\}}(w)}_{g_2(w)} \underbrace{\mathbb{1}_{\{0 \leq u \leq 1\}}(u)}_{g_3(u)}$$

- Check: $\iiint f_{(U,V,W)}(u,v,w)\,du\,dv\,dw = \int_0^{+\infty}\int_0^{+\infty}\int_0^1 v e^{-v-w}\,du\,dv\,dw$

$$= \int_0^{+\infty}\int_0^{+\infty} v e^{-v-w} \underbrace{\left[\int_0^1 du\right]}_{=1} dv\,dw = \int_0^{+\infty} v e^{-v} \underbrace{\left(\int_0^{+\infty} e^{-w}\,dw\right)}_{=1} dv = \int_0^{+\infty} v e^{-v}\,dv$$

$$\overset{IPP}{=} \left[-v e^{-v}\right]_0^{+\infty} + \int_0^{+\infty} e^{-v}\,dv = \left[-e^{-v}\right]_0^{+\infty} = 1.$$

Exercice 4 Let $\alpha > 0$ and $0 < p < 1$....

See exercice 9 Chapter 1, where we have shown that:

- $Y \sim \mathcal{P}(\alpha p)$
- $Z = X - Y \sim \mathcal{P}(\alpha(1-p))$
- we also know that $X \sim \mathcal{P}(\alpha)$, and $Y|X \sim \mathcal{B}(X, p)$

What is the distribution of (Y, Z)? We look for:

$$\mathbb{P}(Y = y, Z = z) \quad \forall (y, z) \in \mathcal{Y} \times \mathcal{Z} \quad \text{where} \quad \begin{cases} \mathcal{Y} = \mathbb{N} \\ \mathcal{Z} = \mathbb{N} \end{cases}$$

$\forall (y, z) \in \mathbb{N}^2$, $\mathbb{P}(Y = y, Z = z) = \mathbb{P}(Y = y, X - Y = z)$

$$= \mathbb{P}(Y = y, X = z + Y) = \mathbb{P}(Y = y, X = z + y)$$

$$= \mathbb{P}(Y = y \mid X = y + z)\, \mathbb{P}(X = y + z)$$

$$= C_{y+z}^{y}\, p^y (1-p)^{y+z-y} \times e^{-\alpha} \frac{\alpha^{y+z}}{(y+z)!} = \frac{\cancel{(y+z)!}}{y!\,(\cancel{y}+z-\cancel{y})!}\, p^y (1-p)^z e^{-\alpha} \frac{\alpha^{y+z}}{\cancel{(y+z)!}}$$

$$= \frac{e^{-\alpha}}{y!\,z!} (\alpha p)^y (\alpha(1-p))^z.$$

Exercice 8 Let X_1, X_2, X_3, X_4 be independent random variables, with $X_1 \sim X_2 \sim X_3 \sim X_4 \sim \mathcal{B}(p)$. Let $M = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, and consider D the determinant of the matrix M. What is $\mathbb{E}[D]$?

A: $\mathbb{E}[D] = \mathbb{E}[X_1 X_4 - X_2 X_3] \overset{\text{linearity}}{=} \mathbb{E}[X_1 X_4] - \mathbb{E}[X_2 X_3] \overset{X_j \perp\!\!\!\perp}{=} \mathbb{E}[X_1]\mathbb{E}[X_4] - \mathbb{E}[X_2]\mathbb{E}[X_3]$

$$= p^2 - p^2 = 0.$$

Exercice 9) We sample two numbers from $\{-2, -1, 0, 1, 2\}$.
Denote by X the product of these two numbers.

(1) What is $\mathbb{E}[X]$ when sampling with replacement?

Denote by X_1 and X_2 the two obtained numbers: we have the following table that summarizes the potential results.

$X_2 \backslash X_1$	-2	-1	0	1	2
-2	4	2	0	-2	-4
-1	2	1	0	-1	-2
0	0	0	0	0	0
1	-2	-1	0	1	2
2	-4	-2	0	2	4

Thus, X_1X_2 takes values in $\{-4; -2; -1; 0; 1; 2; 4\}$.

And $\mathbb{P}(X_1X_2 = -4) = \dfrac{\text{Card}(X_1X_2 = -4)}{\text{Card}(\Omega)} = \dfrac{2}{25}$

$\mathbb{P}(X_1X_2 = -2) = \dfrac{4}{25}$; $\mathbb{P}(X_1X_2 = -1) = \dfrac{2}{25}$; $\mathbb{P}(X_1X_2 = 0) = \dfrac{9}{25}$

$\mathbb{P}(X_1X_2 = 1) = \dfrac{2}{25}$; $\mathbb{P}(X_1X_2 = 2) = \dfrac{4}{25}$; $\mathbb{P}(X_1X_2 = 4) = \dfrac{2}{25}$

$\rightsquigarrow \sum_k \mathbb{P}(X_1X_2 = k) = 1 \rightarrow$ OK!

$\rightsquigarrow \mathbb{E}[X] = \mathbb{E}[X_1X_2] = \cancel{-4 \times \frac{2}{25}} \cancel{- 2 \times \frac{4}{25}} \cancel{- \frac{2}{25}} + 0 \times \frac{9}{25} \cancel{+ 1 \times \frac{2}{25}} \cancel{+ 2 \times \frac{4}{25}} \cancel{+ 4 \times \frac{2}{25}}$

$= 0.$

(2) What is $\mathbb{E}[X]$ when sampling without replacement?

This new experience modifies the outputs that we can obtain.

$X_2 \backslash X_1$	-2	-1	0	1	2
-2	X	2	0	-2	-4
-1	2	X	0	-1	-2
0	0	0	X	0	0
1	-2	-1	0	X	2
2	-4	-2	0	2	X

Here, X_1X_2 takes values in $\{-4;-2;-1;0;2\}$ and

$$\mathbb{P}(X_1X_2=-4)=\frac{2}{20}$$

$$\mathbb{P}(X_1X_2=-2)=\frac{4}{20}$$

$$\mathbb{P}(X_1X_2=-1)=\frac{2}{20}\ ;\ \mathbb{P}(X_1X_2=0)=\frac{8}{20}\ ;\ \mathbb{P}(X_1X_2=2)=\frac{4}{20}$$

then $\mathbb{E}[X]=\mathbb{E}[X_1X_2]=-4\times\frac{2}{20}\cancel{-2\times\frac{4}{20}}-1\times\frac{2}{20}\cancel{+0\times\frac{8}{20}}\cancel{+2\times\frac{4}{20}}$

$$=-\frac{8}{20}-\frac{2}{20}=-\frac{10}{20}=-\frac{1}{2}.$$

Exercice 10 Let X_1 and X_2 two independent r.v., such that

$$X_1\sim\mathcal{B}(p_1) \quad\text{and}\quad X_2\sim\mathcal{B}(p_2).$$

Define $\begin{cases} Y_1=2X_1-1\\ Y_2=2X_2-1\end{cases}$

(a) Is Y_1 independent from Y_2?

(b) Is Y_1 independent from Y_1Y_2?

A: (a). Y_1 takes values in $\{-1;1\}$, as well as Y_2. Let us study

$$\mathbb{P}(Y_1=-1,Y_2=-1)=\mathbb{P}(X_1=0,X_2=0)\overset{X_1\perp\!\!\!\perp X_2}{=}\mathbb{P}(X_1=0)\mathbb{P}(X_2=0)=\mathbb{P}(Y_1=-1)\mathbb{P}(Y_2=-1)$$

$$\mathbb{P}(Y_1=-1,Y_2=1)=\mathbb{P}(X_1=0,X_2=1)=\mathbb{P}(X_1=0)\mathbb{P}(X_2=1)=\mathbb{P}(Y_1=-1)\mathbb{P}(Y_2=1)$$

$$\mathbb{P}(Y_1=1,Y_2=-1)=\mathbb{P}(X_1=1,X_2=0)=\mathbb{P}(X_1=1)\mathbb{P}(X_2=0)=\mathbb{P}(Y_1=1)\mathbb{P}(Y_2=-1)$$

$$\mathbb{P}(Y_1=1,Y_2=1)=\mathbb{P}(X_1=1,X_2=1)=\mathbb{P}(X_1=1)\mathbb{P}(X_2=1)=\mathbb{P}(Y_1=1)\mathbb{P}(Y_2=1)$$

Therefore $\forall x,y\in\{0;1\}^2$, $\mathbb{P}(Y_1=x,Y_2=y)=\mathbb{P}(Y_1=x)\mathbb{P}(Y_2=y)\Rightarrow Y_1\perp\!\!\!\perp Y_2$.

(b) $\begin{cases} Y_1 \text{ takes values in } \{-1;1\} \\ Y_2 \quad " \quad " \quad " \quad \{-1;1\} \\ Y_1Y_2 \quad " \quad " \quad " \quad \{-1;1\} \end{cases}$

- $\mathbb{P}(Y_1Y_2=-1) = \mathbb{P}(\{Y_1=-1, Y_2=1\} \cup \{Y_1=1, Y_2=-1\}) = \mathbb{P}(Y_1=-1, Y_2=+1) + \mathbb{P}(Y_1=1, Y_2=-1)$

$\overset{Y_1 \perp\!\!\!\perp}{=} \mathbb{P}(Y_1=-1)\mathbb{P}(Y_2=1) + \mathbb{P}(Y_1=1)\mathbb{P}(Y_2=-1) = \mathbb{P}(X_1=0)\mathbb{P}(X_2=1) + \mathbb{P}(X_1=1)\mathbb{P}(X_2=0)$

$= (1-p_1)p_2 + p_1(1-p_2) = p_1 + p_2 - 2p_1p_2.$

- $\mathbb{P}(Y_1Y_2=1) = \mathbb{P}(\{Y_1=1, Y_2=1\} \cup \{Y_1=-1, Y_2=-1\})$

$= \mathbb{P}(Y_1=1, Y_2=1) + \mathbb{P}(Y_1=-1, Y_2=-1) = \mathbb{P}(Y_1=1)\mathbb{P}(Y_2=1) + \mathbb{P}(Y_1=-1)\mathbb{P}(Y_2=-1)$

$= \mathbb{P}(X_1=1)\mathbb{P}(X_2=1) + \mathbb{P}(X_1=0)\mathbb{P}(X_2=0) = p_1p_2 + (1-p_1)(1-p_2)$

$= p_1p_2 + 1 - p_2 - p_1 + p_1p_2 = 1 + 2p_1p_2 - p_1 - p_2.$

- $\mathbb{P}(Y_1Y_2=-1, Y_1=-1) = \mathbb{P}(Y_1Y_2=-1 \mid Y_1=-1)\mathbb{P}(Y_1=-1)$

$= \mathbb{P}(Y_2=1 \mid Y_1=-1)\mathbb{P}(Y_1=-1) \overset{Y_1 \perp\!\!\!\perp Y_2}{=} \mathbb{P}(Y_2=1)\mathbb{P}(Y_1=-1)$

$= \mathbb{P}(X_2=1)\mathbb{P}(X_1=0) = p_2(1-p_1) = p_2 - p_1p_2$

- $\mathbb{P}(Y_1Y_2=-1)\mathbb{P}(Y_1=-1) = (p_1+p_2-2p_1p_2)\mathbb{P}(X_1=0) = (p_1+p_2-2p_1p_2)(1-p_1)$

$= p_1 - p_1^2 + p_2 - p_1p_2 - 2p_1p_2 + 2p_1^2p_2 = p_1^2(2p_2-1) - 3p_1p_2 + p_1 + p_2.$

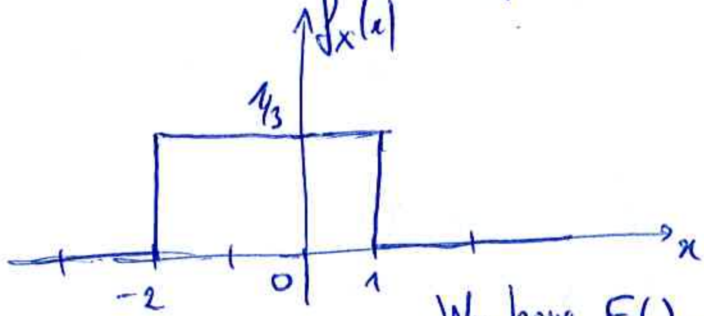
Therefore $\mathbb{P}(Y_1Y_2=-1, Y_1=-1) \neq \mathbb{P}(Y_1Y_2=-1)\mathbb{P}(Y_1=-1)$ for some p_1, p_2

Hence $Y_1Y_2 \not\!\perp\!\!\!\perp Y_1$

Exercice 11 Let $X \sim \mathcal{U}([-2,1])$ follow a continuous Uniform distribution. Define $Y = |X|$, $Z = \max(X, 0)$.

Q: What are the cdf of Y and Z? Do they admit a density function? Are they independent?

A: First, remind that if $X \sim \mathcal{U}([\overset{a}{-2}, \overset{b}{1}])$, then the density looks like:



$$f_X(x) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(x) = \frac{1}{3} \mathbb{1}_{[-2,1]}(x)$$

Indeed, $\int_{-\infty}^{+\infty} f_X(x)dx = \int_{-2}^{1} \frac{1}{3} dx = \frac{1}{3}\left[x\right]_{-2}^{1} = 1.$

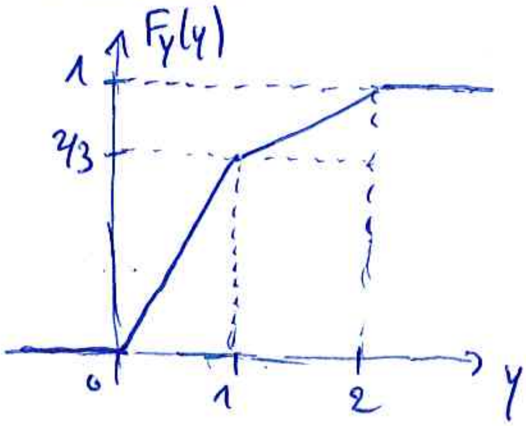
We have $F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{else} \end{cases} \Rightarrow F_X(x) = \begin{cases} 0 & \text{if } x < -2 \\ \frac{x+2}{3} & \text{if } -2 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

- Let $Y = |X|$:

$$\forall y \geq 0, \; F_Y(y) = \mathbb{P}(Y \le y)$$
$$= \mathbb{P}(|X| \le y) = \mathbb{P}(-y \le X \le y) = F_X(y) - F_X(-y)$$

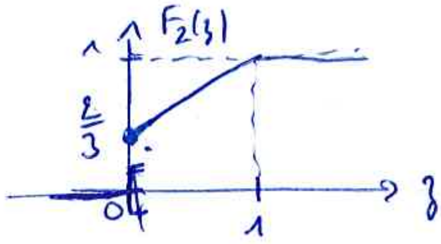
$$= \begin{cases} 0 & \text{if } y < 0 \\ \frac{y+2}{3} - \frac{-y+2}{3} & \text{if } 0 \le y \le 1 \\ 1 - \frac{-y+2}{3} & \text{if } 1 \le y \le 2 \\ 1 & \text{if } y > 2 \end{cases} = \begin{cases} 0 & \text{if } y < 0 \\ \frac{2y}{3} & \text{if } 0 \le y \le 1 \\ \frac{1}{3} + \frac{y}{3} & \text{if } 1 \le y \le 2 \\ 1 & \text{if } y > 2 \end{cases}$$

Illustration:



Yes, $|X|$ admits a density function, since $F_{|X|}$ is continuously increasing and admits a derivative.

- Let $Z = \max(0,X)$:



$$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(\max(0,X) \leq z)$$

$$= \mathbb{P}(0 \leq z, X \leq z) = \begin{cases} 0 & \text{if } z < 0 \\ F_X(z) & \text{if } z \geq 0 \end{cases} = \begin{cases} 0 & \text{if } z < 0 \\ \frac{z+2}{3} & \text{if } 0 \leq z \leq 1 \\ 1 & \text{if } z > 1 \end{cases}$$

$\Rightarrow$ Z is càdlàg and does not admit a density function as it is not continuously increasing, with no derivative on 0.

- $Y \perp\!\!\!\perp Z$? Remind that $F_Y(z) = \begin{cases} 0 & \text{if } z < 0 \\ \frac{2z}{3} & \text{if } 0 \leq z \leq 1 \\ \frac{1}{3} + \frac{z}{3} & \text{if } 1 \leq z \leq 2 \\ 1 & \text{if } z > 2 \end{cases}$

If we consider $z \in [1;2]$,
then $\mathbb{P}(Z \leq z) = 1$

$\mathbb{P}(Y \leq z) = \frac{1}{3} + \frac{1}{3} z$

and clearly $\mathbb{P}(Y \leq z, Z \leq z) \neq \mathbb{P}(Y \leq z) \underbrace{\mathbb{P}(Z \leq z)}_{=1} = \frac{1}{3} + \frac{1}{3} z$

Exercice 12 Let (X,Y) be the two-dimensional random vector with values in $\{-2;0;1\} \times \{-\frac{1}{2};0;1\}$. Its distribution is given by the following joint distribution:

$y \backslash x$	-2	0	1
$-\frac{1}{2}$	$1/10$	α	0
0	$3/10$	0	$3/10$
1	$1/10$	$1/10$	0

1) Give the value of α:

α is such that:

$$\sum_x \sum_y \mathbb{P}(X=x, Y=y) = 1.$$

We thus have $\frac{1}{10} + \alpha + 0 + \frac{3}{10} + 0 + \frac{3}{10} + \frac{1}{10} + \frac{1}{10} + 0 = 1$

$\alpha + \frac{9}{10} = 1 \Rightarrow \alpha = \frac{1}{10}$

2) What is the distribution of Y?

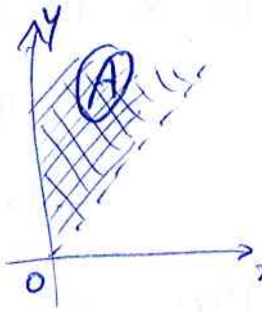
Y has a discrete distribution, with values in $\{-\frac{1}{2}; 0; 1\}$.

- $\mathbb{P}(Y=-\frac{1}{2}) = \mathbb{P}(Y=-\frac{1}{2}, X=-2) + \mathbb{P}(Y=-\frac{1}{2}, X=0) + \mathbb{P}(Y=-\frac{1}{2}, X=1)$
  $= 2/10$
- $\mathbb{P}(Y=0) = 6/10$
- $\mathbb{P}(Y=1) = 2/10$.

3) Is Y independent from X?

No, since $\left.\begin{array}{l}\mathbb{P}(X=0) = 2/10 \\ \mathbb{P}(Y=0) = 6/10\end{array}\right\} \Rightarrow \mathbb{P}(X=0)\,\mathbb{P}(Y=0) = \frac{12}{100} \neq \mathbb{P}(X=0, Y=0) = 0$

Exercice 13 Let (X,Y) be the two-dimensional random vector with density function: $f_{(X,Y)}(x,y) = e^{-y}\,\mathbb{1}_{\{0 \le x \le y\}}$



1) Check that f is a density function:

- $f_{(X,Y)}(x,y) \geq 0 \quad \forall (x,y)$
- f is continuous
- $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-y}\,\mathbb{1}_{\{0\le x\le y\}}\,dx\,dy = \int_0^{+\infty}\left(\int_x^{+\infty} e^{-y}\,dy\right)dx$

$$= \int_0^{+\infty}\left[-e^{-y}\right]_x^{+\infty} dx = \int_0^{+\infty} e^{-x}\,dx = \left[-e^{-x}\right]_0^{+\infty} = 1.$$

2) Is X independent from Y? We have $\mathbb{P}(Y \le 1, X \geq 2) = 0$ since there is no density (no probability mass) when $x \geq y$.

And $\mathbb{P}(Y \le 1) = \displaystyle\int_0^1\left(\int_0^y e^{-y}\,dx\right)dy = \int_0^1 e^{-y}\left(\int_0^y dx\right)dy > 0$

$\mathbb{P}(X \geq 2) = \displaystyle\int_{2}^{+\infty} e^{-y}\int_0^y dx\,dy > 0$

Hence, $\mathbb{P}(Y \leq 1, X \geq 2) \neq \mathbb{P}(Y \leq 1)\, \mathbb{P}(X \geq 2)$

$\Rightarrow$ Y is not independent from X.

(3) Are X and $Y-X$ independent random variables ?

We will answer thanks to the method "fonction muette": Let h be a positive measurable function. then

$$\mathbb{E}[h(U,V)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u,v)\, f_{(U,V)}(u,v)\, du\, dv \text{, with } \begin{cases} U = X \\ V = Y-X \end{cases} \Leftrightarrow \begin{cases} X = U \\ Y = V+U \end{cases}$$

In the same way, $\mathbb{E}[h(X,Y-X)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x,y-x)\, f_{(X,Y)}(x,y)\, dx\, dy$

Let ϕ the diffeomorphism such that:

$$\phi : \mathbb{R}^{+2} \longrightarrow \mathbb{R}^{+2} \qquad \Leftrightarrow \qquad \phi^{-1} : \mathbb{R}^{+2} \longrightarrow \mathbb{R}^{+2}$$
$$(x,y) \longmapsto (u,v) = (x, y-x) \qquad\qquad (u,v) \longmapsto (x,y) = (u, u+v)$$

Hence $J_{\phi^{-1}}(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow |\det J_{\phi^{-1}}| = 1$

We thus obtain: $\mathbb{E}[h(U,V)] = \mathbb{E}[h(X,Y-X)]$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u,v)\, \underbrace{|\det J_{\phi^{-1}}|}_{\text{comes from } dx dy \text{ that became } du dv}\, e^{-\overset{y}{\overset{\shortparallel}{(u+v)}}}\, \mathbb{1}_{\{u \geq 0, v \geq 0\}}\, du\, dv$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} h(u,v)\, \underbrace{e^{-u}\, \mathbb{1}_{\{u \geq 0\}}\, e^{-v}\, \mathbb{1}_{\{v \geq 0\}}}_{f_{(U,V)}(u,v) = f_U(u)\, f_V(v)}\, du\, dv.$$

$\Rightarrow$ X and $Y-X$ are therefore independent, and $X \sim Y-X \sim \mathcal{E}xp(1)$.

(4) What is the distribution of $\left(Y-X, \frac{X}{Y}\right)$?

With the same methodology: $\phi : \mathbb{R}^{+2} \longrightarrow \mathbb{R}^+ \times [0,1]$

$$(x,y) \longmapsto (u,v) = \left(y-x, \frac{x}{y}\right)$$

$$\begin{cases} U = Y - X \\ V = \frac{X}{Y} \end{cases} \Leftrightarrow \begin{cases} U = Y - VY \\ X = VY \end{cases} \Leftrightarrow \begin{cases} U = Y(1-V) \\ X = VY \end{cases} \Leftrightarrow \begin{cases} Y = U/(1-V) \\ X = V\frac{U}{1-V} \end{cases}$$

$$\Rightarrow \begin{cases} X = \frac{UV}{1-V} \\ Y = \frac{U}{1-V} \end{cases} \Rightarrow \begin{array}{l} \phi^{-1} : \mathbb{R}^+ \times [0,1] \longrightarrow \mathbb{R}^{+2} \\ (u,v) \longmapsto (x,y) = \left(\frac{uv}{1-v}, \frac{u}{1-v}\right) \end{array}$$

thus $J_{\phi^{-1}}(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{v}{1-v} & \frac{u}{(1-v)^2} \\ \frac{1}{1-v} & \frac{u}{(1-v)^2} \end{pmatrix}$

$$\Rightarrow \det J_{\phi^{-1}}(u,v) = \frac{v}{1-v} \times \frac{u}{(1-v)^2} - \frac{1}{1-v}\frac{u}{(1-v)^2} = \frac{uv}{(1-v)^3} - \frac{u}{(1-v)^3}$$

$$= \frac{u}{(1-v)^3}(v-1) = -\frac{u}{(1-v)^2}$$

$$\Rightarrow \mathbb{E}[h(U,V)] = \mathbb{E}\left[h\left(Y-X, \frac{X}{Y}\right)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h\left(y-x, \frac{x}{y}\right) f_{(X,Y)}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u,v) \left| -\frac{u}{(1-v)^2} \right| e^{-\frac{u}{1-v}} \mathbb{1}_{\{u \geq 0,\, 0 \leq v \leq 1\}}(u,v)\, du\, dv$$

We obtain $f_{(U,V)}(u,v) = \frac{u}{(1-v)^2} e^{-\frac{u}{1-v}} \mathbb{1}_{\{u \geq 0,\, 0 \leq v \leq 1\}}(u,v)$

That cannot be seperated into some product $f_U(u) \times f_V(v) \Rightarrow$ not independent!

Exercice 14 Use the results about the generating or characteristic functions, see Chapter 1.

Exercice 15 Two <u>independent</u> factories receive each day the respective random numbers X and Y of calls. We have $\begin{cases} X \sim \mathcal{P}(\lambda) \\ Y \sim \mathcal{P}(\mu) \end{cases}$.

(1) What is the probability that the cumulated number of received calls would not exceed 3, with $\lambda = 2$ and $\mu = 4$ (in one day)?

$\mathbb{P}(X+Y \leq 3) = ?$

We already know that $\left.\begin{array}{l} X \sim \mathcal{P}(\lambda) \\ Y \sim \mathcal{P}(\mu) \\ X \perp\!\!\!\perp Y \end{array}\right\} \Rightarrow X+Y \sim \mathcal{P}(\lambda+\mu)$

Hence $\mathbb{P}(X+Y \leq 3) = \sum_{k=0}^{3} \mathbb{P}(X+Y=k) = \sum_{k=0}^{3} e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^k}{k!} = \ldots$

(2) What is the probability that "$X=k$" knowing that "$X+Y=n$", for integers k and n. ($k \leq n$)?

$$\mathbb{P}(X=k \mid X+Y=n) = \frac{\mathbb{P}(X=k, X+Y=n)}{\mathbb{P}(X+Y=n)} = \frac{\mathbb{P}(X=k, Y=n-k)}{\mathbb{P}(X+Y=n)} \overset{X \perp\!\!\!\perp Y}{=} \frac{\mathbb{P}(X=k)\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)}$$

$$= \frac{\cancel{e^{-\lambda}} \times \frac{\lambda^k}{k!} \times \cancel{e^{-\mu}} \frac{\mu^{n-k}}{(n-k)!}}{\cancel{e^{-(\lambda+\mu)}} \frac{(\lambda+\mu)^n}{n!}} = \frac{n!}{(\lambda+\mu)^n} \frac{\lambda^k}{k!} \frac{\mu^{n-k}}{(n-k)!} = \frac{n!}{k!(n-k)!} \frac{\lambda^k \mu^{n-k}}{(\lambda+\mu)^{n-k+k}}$$

$$= C_n^k \left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{n-k} \Rightarrow X \mid X+Y=n \sim \mathcal{B}in\left(n, p=\frac{\lambda}{\lambda+\mu}\right)$$

(3) Numerical application!

Exercice 16 Suppose that (X,Y) is a random vector with the following joint distribution:

$X \backslash Y$	0	1	2
0	$1/9$	$2/9$	0
1	0	$1/9$	$2/9$
2	$2/9$	0	$1/9$

(1) Show that $Cov(X,Y)=0$, but that $X \not\perp\!\!\!\perp Y$:

$Cov(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$.

- Distribution of X: $\mathbb{P}(X=0) = \mathbb{P}(X=0, Y=0) + \mathbb{P}(X=0, Y=1) + \mathbb{P}(X=0, Y=2)$
$= 3/9$

$\mathbb{P}(X=1) = 3/9$

$\mathbb{P}(X=2) = 3/9$ $\Rightarrow \mathbb{E}[X] = 0 \times \frac{3}{9} + 1 \times \frac{3}{9} + 2 \times \frac{3}{9} = 1$

- Distribution of Y: $\left.\begin{array}{l}\mathbb{P}(Y=0) = 3/9 \\ \mathbb{P}(Y=1) = 3/9 \\ \mathbb{P}(Y=2) = 3/9\end{array}\right\} \Rightarrow \mathbb{E}[Y] = 1.$

then $Cov(X,Y) = \mathbb{E}[(X-1)(Y-1)] = \mathbb{E}[XY - X - Y + 1] = \mathbb{E}[XY] - \mathbb{E}[X] - \mathbb{E}[Y] + 1$

$= \mathbb{E}[XY] - 1 - \cancel{1} + \cancel{1} = \mathbb{E}[XY] - 1.$

- $\mathbb{E}[XY] = ?$ XY takes values in $\{0;1;2;4\}$.

$\mathbb{P}(XY=0) = \mathbb{P}(X=0, Y=0) + \mathbb{P}(X=0, Y=1) + \mathbb{P}(X=0, Y=2) + \mathbb{P}(X=1, Y=0) + \mathbb{P}(X=2, Y=0)$
$= 1/9 + 2/9 + 0 + 0 + 2/9 = 5/9$

$\mathbb{P}(XY=1) = \mathbb{P}(X=1, Y=1) = 1/9$

$\mathbb{P}(XY=2) = \mathbb{P}(X=1, Y=2) + \mathbb{P}(X=2, Y=1) = 2/9 + 0 = 2/9$

$\mathbb{P}(XY=4) = \mathbb{P}(X=2, Y=2) = 1/9$

$\sum \mathbb{P} = 1$

Thus $\mathbb{E}[XY] = 0 \times \frac{5}{9} + 1 \times \frac{1}{9} + 2 \times \frac{2}{9} + 4 \times \frac{1}{9} = 1.$

We obtain at the end: $Cov(X,Y) = \mathbb{E}[XY] - 1 = 0$.

However, $\left.\begin{array}{l}\mathbb{P}(X=1) = 1/3 \\ \mathbb{P}(Y=0) = 1/3\end{array}\right\} \Rightarrow \mathbb{P}(X=1)\mathbb{P}(Y=0) = \frac{1}{9} \neq \mathbb{P}(X=1, Y=0) = 0 \Rightarrow X \not\perp\!\!\!\perp Y$

(2) What are the generating functions of $X, Y, X+Y$; check that $G_X G_Y = G_{X+Y}$.

- $G_X(t) = \mathbb{E}[t^X] = \sum_{k=0}^{2} t^k \mathbb{P}(X=k) = t^0 \mathbb{P}(X=0) + t\,\mathbb{P}(X=1) + t^2 \mathbb{P}(X=2)$
$= 1/3 + 1/3\,t + 1/3\,t^2 = \frac{1}{3}(1+t+t^2)$
- $G_Y(t) = \mathbb{E}[t^Y] = G_X(t)$.
- $X \not\perp\!\!\!\perp Y \Rightarrow$ in full generality we should not have $G_{X+Y}(t) = G_X(t)G_Y(t)$.

However, here $X+Y$ takes values in $\{0;1;2;3;4\}$, and

$\mathbb{P}(X+Y=0) = \mathbb{P}(X=0, Y=0) = 1/9$

$\mathbb{P}(X+Y=1) = \mathbb{P}(X=0, Y=1) + \mathbb{P}(X=1, Y=0) = 2/9$

$\mathbb{P}(X+Y=2) = \mathbb{P}(X=0, Y=2) + \mathbb{P}(X=1, Y=1) + \mathbb{P}(X=2, Y=0) = 3/9$

$\mathbb{P}(X+Y=3) = \mathbb{P}(X=1, Y=2) + \mathbb{P}(X=2, Y=1) = 2/9$

$\mathbb{P}(X+Y=4) = \mathbb{P}(X=2, Y=2) = 1/9$

$\Rightarrow G_{X+Y}(t) = \mathbb{E}[t^{X+Y}] = \sum_{k=0}^{4} t^k \mathbb{P}(X+Y=k) = \frac{t^0}{9} + \frac{2t}{9} + \frac{3t^2}{9} + \frac{2t^3}{9} + \frac{t^4}{9}$

And $(G_X(t))^2 = \left(\frac{1}{3}(1+t+t^2)\right)^2 = \frac{1}{9}\left(1+t^2+t^4+2t+2t^2+2t^3\right) = \frac{1}{9} + \frac{2t}{9} + \frac{3t^2}{9} + \frac{2t^3}{9} + \frac{t^4}{9}$.

We thus have $G_X(t)G_Y(t) = G_{X+Y}(t)$ although $X \not\perp\!\!\!\perp Y$!

(3) Compute $\mathbb{E}[X|Y]$:

$\mathbb{E}[X|Y]$ is a random variable, where Y can take values in $\{0;1;2\}$.

- $\mathbb{E}[X|Y=0] = \sum_{k=0}^{2} k\,\mathbb{P}(X=k|Y=0) = 0\,\mathbb{P}(X=0|Y=0) + \mathbb{P}(X=1|Y=0) + 2\,\mathbb{P}(X=2|Y=0)$

$= \frac{\mathbb{P}(X=1, Y=0)}{\mathbb{P}(Y=0)} + 2\frac{\mathbb{P}(X=2, Y=0)}{\mathbb{P}(Y=0)} = 0 + 2\frac{2/9}{1/3} = \frac{4}{3}$

- In the same way, $\bullet\ \mathbb{E}[X|Y=1] = 1/3$

$\bullet\ \mathbb{E}[X|Y=2] = \frac{4}{3}$

Exercice 17 Let X be a real-valued random variable with symmetric distribution (X and $-X$ have the same distribution).
Let $\mathcal{E}$ be an independent r.v. from X, such that $\begin{cases} \mathbb{P}(\mathcal{E}=1)=p \\ \mathbb{P}(\mathcal{E}=-1)=1-p \end{cases}$, $p\in]0,1[$.

(1)- What is the distribution of $\mathcal{E}X$?

A: $F_{\mathcal{E}X}(t) = \mathbb{P}(\mathcal{E}X\leq t) = \mathbb{P}(\mathcal{E}X\leq t, \mathcal{E}=-1) + \mathbb{P}(\mathcal{E}X\leq t, \mathcal{E}=1)$

$= \mathbb{P}(\mathcal{E}X\leq t \mid \mathcal{E}=-1)\,\mathbb{P}(\mathcal{E}=-1) + \mathbb{P}(\mathcal{E}X\leq t\mid \mathcal{E}=1)\,\mathbb{P}(\mathcal{E}=1)$

$= (1-p)\,\mathbb{P}(-X\leq t) + p\,\mathbb{P}(X\leq t)$, by assumption $X\sim -X$

$= (1-p)\,\mathbb{P}(X\leq t) + p\,\mathbb{P}(X\leq t) = \mathbb{P}(X\leq t) = F_X(t)$.

$\Rightarrow$ X and $\mathcal{E}X$ have the same distribution.

(2)- Which condition on p ensures that $\mathrm{Cov}(X,\mathcal{E}X)=0$? In this case, are $\mathcal{E}X$ and X independent?

A: $\mathrm{Cov}(X,\mathcal{E}X) = \mathbb{E}\big[(X-\mathbb{E}[X])(\mathcal{E}X-\mathbb{E}[\mathcal{E}X])\big]$;

We know that $\begin{cases} X \text{ has a symmetric distribution} \Rightarrow \mathbb{E}[X]=0 \\ \mathcal{E}X \text{ has the same distribution as } X \Rightarrow \mathbb{E}[\mathcal{E}X]=0 \end{cases}$

Therefore $\mathrm{Cov}(X,\mathcal{E}X) = \mathbb{E}[X\mathcal{E}X] = \mathbb{E}[\mathcal{E}X^2] \overset{\mathcal{E}\perp\!\!\!\perp X}{=} \mathbb{E}[\mathcal{E}]\,\mathbb{E}[X^2]$

$= \big(-1(1-p) + 1\times p\big)\,\mathrm{Var}(X) = (2p-1)\underbrace{\mathrm{Var}(X)}_{\neq 0} \Rightarrow \boxed{p=\frac{1}{2}}$.

- Consider $p=\frac{1}{2}$; $X \perp\!\!\!\perp \mathcal{E}X$?

We know that $X\perp\!\!\!\perp Y \Rightarrow \mathrm{Cov}(X,Y)=0$, thus $\mathrm{Cov}(X,Y)\neq 0 \Rightarrow X \not\perp\!\!\!\perp Y$.

Introduce f and g two bounded measurable functions, and study

$\mathrm{Cov}(f(X), g(\mathcal{E}X))$. Take for instance $f(x)=g(x)=|x|$.

$= \mathbb{E}[f(X)g(\mathcal{E}X)] - \mathbb{E}[f(X)]\,\mathbb{E}[g(\mathcal{E}X)]$

Hence, $\mathrm{Cov}(|X|, |\varepsilon X|) = \mathbb{E}[|X||\varepsilon X|] - \mathbb{E}[|X|]\mathbb{E}[|X|]$

$$= \mathbb{E}[|X|^2] - (\mathbb{E}[|X|])^2 = \mathrm{Var}(|X|) \quad (\mathrm{I})$$

- If this covariance equals 0, then it means that $|X| = c$ almost surely (a.s.).
Then $X = c$ or $X = -c$ a.s.
But X has a symetric distribution thus $\mathbb{P}(X=c) = \mathbb{P}(X=-c) = 1/2$.
However, having no covariance does not guarantee independence!
- Take a counter-example: $X \sim \mathcal{N}(0,1)$ (symetric distribution) ($\varepsilon X \sim \mathcal{N}(0,1)$).
Then $\mathrm{Var}(|X|) \neq 0$, then $\mathrm{Cov}(|X|, |\varepsilon X|) \neq 0$ from (I)
$\Rightarrow X \not\perp\!\!\!\perp \varepsilon X$.

(3) - Let $Y = \mathbb{1}_{X>0} - \mathbb{1}_{X<0}$. Give the distributions of Y and XY.
What is $\mathrm{Cov}(|X|, Y)$? Are $|X|$ et Y independent?

- Y takes values in $\{-1; 0; 1\}$: $\begin{cases} \mathbb{P}(Y=-1) = \mathbb{P}(X<0) \overset{X \text{ symetric}}{=} \mathbb{P}(X>0) = \alpha \\ \mathbb{P}(Y=1) = \mathbb{P}(X>0) = \alpha \end{cases}$

Therefore $\mathbb{P}(Y=0) = 1 - (\mathbb{P}(Y=-1) + \mathbb{P}(Y=1))$
$= 1 - 2\alpha$

- $F_{XY}(t) = \mathbb{P}(XY \leq t) = \mathbb{P}(XY \leq t, Y=-1) + \mathbb{P}(XY \leq t, Y=0) + \mathbb{P}(XY \leq t, Y=1)$

$$= \mathbb{P}(XY \leq t | Y=-1)\mathbb{P}(Y=-1) + \mathbb{P}(XY \leq t | Y=0)\mathbb{P}(Y=0) + \mathbb{P}(XY \leq t | Y=1)\mathbb{P}(Y=1)$$

$$= \mathbb{P}(-X \leq t) \times \alpha + \mathbb{P}(0 \leq t)(1-2\alpha) + \mathbb{P}(X \leq t) \times \alpha$$

$$= 2\alpha F_X(t) + (1-2\alpha)\mathbb{P}(0 \leq t)$$

$$= \begin{cases} 2\alpha F_X(t) & \text{if } t<0 \\ (1-2\alpha) + 2\alpha F_X(t) & \text{if } t \geq 0 \end{cases}$$

Notice that $XY = \begin{cases} +X & \text{if } X>0 \\ -X & \text{if } X<0 \\ 0 & \text{if } X=0 \end{cases} \Rightarrow XY = |X|$. (10)

- $$\begin{aligned} \mathrm{Cov}(|X|, Y) &= \mathrm{Cov}(XY, Y) \\ &= \mathbb{E}\big[(XY - \mathbb{E}[XY])(Y - \mathbb{E}[Y])\big], \text{ with } \mathbb{E}[Y]=0 \\ &= \mathbb{E}\big[(XY - \mathbb{E}[XY])Y\big] = \mathbb{E}\big[XY^2 - Y\,\mathbb{E}[XY]\big] \\ &= \mathbb{E}[XY^2] - \mathbb{E}\big[Y\,\mathbb{E}[XY]\big] = \mathbb{E}[XY^2] - \mathbb{E}[XY]\underbrace{\mathbb{E}[Y]}_{=0} \\ &= \mathbb{E}[XY^2] \neq 0 \Rightarrow |X| \not\perp\!\!\!\perp Y. \end{aligned}$$

Exercise 18 Let X be a random vector, Gaussianly distributed such that:

$$X \sim \mathcal{N}_3(0, \Gamma) \quad \text{where} \quad \Gamma = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Q: Find a vector $a(X)$ where a is a linear application from $\mathbb{R}^3$ to $\mathbb{R}^3$, with independent components.

A: We can see from Γ the variance-covariance matrix that this gaussian vector has the following properties: $\begin{cases} \mathrm{Var}(X_1) = 3 = \mathrm{Var}(X_2) \\ \mathrm{Var}(X_3) = 2 \qquad \mathrm{Cov}(X_1,X_3)=0 \\ \mathrm{Cov}(X_1,X_2) = -1 \qquad \mathrm{Cov}(X_2,X_3)=0 \\ \mathbb{E}[X_1] = \mathbb{E}[X_2] = \mathbb{E}[X_3] = 0. \end{cases}$

We thus know that $\begin{cases} X_1 \perp\!\!\!\perp X_3 \\ X_2 \perp\!\!\!\perp X_3 \\ X_1 \not\perp\!\!\!\perp X_2 \end{cases}$

We also know that $X, Y \perp\!\!\!\perp Z$ implies $aX + bY \perp\!\!\!\perp Z \quad \forall (a,b) \in \mathbb{R}^2$.

Hence we look for (a,b) such that $\mathrm{Cov}(aX_1 + bX_2, X_2) = 0$.

Then $\mathrm{Cov}(aX_1 + bX_2, X_2) = a\,\mathrm{Cov}(X_1,X_2) + b\,\mathrm{Var}(X_2) = 3b - a = 0 \Rightarrow a = 3b$.

We thus propose $a: \mathbb{R}^3 \to \mathbb{R}^3$

$$X = (X_1, X_2, X_3) \longmapsto X' = \begin{cases} X_1' = 3X_1 + X_2 \\ X_2' = X_2 \end{cases} \Rightarrow \underline{\forall (i,j)\ \mathrm{Cov}(X_i', X_j') = 0}$$

All the components of X' are now independent, and

$$\alpha(X) = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} X_1' \\ X_2' \\ X_3' \end{pmatrix}$$

Exercice 19 Let $X \sim \mathcal{N}(0,1)$, and $\varepsilon \perp\!\!\!\perp X$ such that $\begin{cases} \mathbb{P}(\varepsilon = -1) = 1/2 \\ \mathbb{P}(\varepsilon = 1) = 1/2 \end{cases}$.

Q: • What is the distribution of $Y = \varepsilon X$?
• Is (X,Y) a gaussian vector?
• Compute $\mathrm{Cov}(X,Y)$.

A: • $F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}(\varepsilon X \leq x) = \mathbb{P}(\varepsilon X \leq x, \varepsilon = -1) + \mathbb{P}(\varepsilon X \leq x, \varepsilon = 1)$

$= \mathbb{P}(\varepsilon X \leq x \mid \varepsilon = -1)\mathbb{P}(\varepsilon = -1) + \mathbb{P}(\varepsilon X \leq x \mid \varepsilon = 1)\mathbb{P}(\varepsilon = 1)$

$= \mathbb{P}(-X \leq x)\,\frac{1}{2} + \mathbb{P}(X \leq x) \times \frac{1}{2} = \frac{1}{2}\mathbb{P}(X \geq -x) + \frac{1}{2}\mathbb{P}(X \leq x)$

$\overset{X \sim \mathcal{N}(0,1)}{=} 2 \times \frac{1}{2}\mathbb{P}(X \leq x) = F_X(x)$, thus $Y \sim \mathcal{N}(0,1)$.

• If (X,Y) were gaussian vector, then $\forall (a,b) \in \mathbb{R}^2$, $aX + bY \sim \mathcal{N}(\ ,\ )$.
Here this means that $aX + b\varepsilon X$ for any (a,b) should be gaussian.
Taking $(a,b) = (1,1)$, then consider the random variable $X + \varepsilon X$.

$\mathbb{P}(X + \varepsilon X = 0) = \mathbb{P}(\varepsilon = -1) = \frac{1}{2} \neq 0$, and if $X + \varepsilon X$ were gaussian, we would have $\mathbb{P}(X + \varepsilon X = 0) = 0 \Rightarrow X + \varepsilon X \not\sim \mathcal{N}(\ ,\ )$.

$\Rightarrow$ This is not a gaussian vector.

• $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, \varepsilon X) = \mathbb{E}[X \varepsilon X] - \underbrace{\mathbb{E}[X]}_{=0}\underbrace{\mathbb{E}[\varepsilon X]}_{=0} = \mathbb{E}[X^2 \varepsilon]$

$= \mathbb{E}[X^2 \mathbb{1}_{\varepsilon = 1} + (-1) X^2 \mathbb{1}_{\varepsilon = -1}] = \mathbb{E}[X^2 \mathbb{1}_{\varepsilon = 1} - X^2 \mathbb{1}_{\varepsilon = -1}]$

$$= \mathbb{E}[X^2]\,\mathbb{P}(\varepsilon = 1) - \mathbb{E}[X^2]\,\mathbb{P}(\varepsilon = -1) = 0.$$

Exercice 20

(1) Prove that the matrix $\Gamma = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$ is a covariance matrix $\Leftrightarrow r \in [-1,1]$.

$\begin{cases} \det(\Gamma) = 1 - r^2 \\ \text{Trace}(\Gamma) = 2 \\ \Gamma \text{ is symmetric.} \end{cases}$ Denoting by (λ_1, λ_2) the eigen values of Γ, we have $\begin{cases} Tr(\Gamma) = \lambda_1 + \lambda_2 \\ \det(\Gamma) = \lambda_1 \lambda_2 \end{cases}$

We thus get:

$\det(\Gamma) = \lambda_1\lambda_2 = 1 - r^2 \Rightarrow \det(\Gamma) \geq 0 \Leftrightarrow \begin{cases} \lambda_1 \text{ and } \lambda_2 \geq 0 \\ \text{or} \\ \lambda_1 \text{ and } \lambda_2 \leq 0 \end{cases}$ and $r \in [-1,1]$

But $Tr(\Gamma) = 2 = \lambda_1 + \lambda_2 \Rightarrow \lambda_1$ and $\lambda_2 \geq 0$

Therefore this matrix Γ is symmetric, and positive definite $\Rightarrow$ this is a variance-covariance matrix.

(2) In what follows, we suppose that this condition is fulfilled, and we consider (X,Y) a gaussian vector such that $(X,Y) \sim \mathcal{N}_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \Gamma\right)$.

Q: Find a_0 the value of a that minimizes $\mathbb{E}[(Y-aX)^2]$.

A: $\mathbb{E}[(Y-aX)^2] = \mathbb{E}[Y^2 - 2aXY + a^2X^2] = \mathbb{E}[Y^2] - 2a\,\mathbb{E}[XY] + a^2\mathbb{E}[X^2]$

$$= Var(Y) + (\underbrace{\mathbb{E}[Y]}_{=0})^2 - 2a\,\mathbb{E}[XY] + a^2\left(Var(X) + (\underbrace{\mathbb{E}[X]}_{=0})^2\right)$$

$$= Var(Y) - 2a\,Cov(X,Y) + a^2\,Var(X)$$

$$= 1 - 2a\,Cov(X,Y) + a^2 \Rightarrow \frac{d}{da}\mathbb{E}[(Y-aX)^2] = 2a - 2\underbrace{Cov(X,Y)}_{=r}$$

Given that $\mathbb{E}[(Y-aX)^2]$ is a convex function, the null of the derivative corresponds to a minimum $\Rightarrow 2a_0 - 2r = 0 \Rightarrow \boxed{a_0 = r}$

The covariance between Y and X is the slope that minimizes the mean squared error between aX and Y. (well-known in linear regression).

(3) Give the distributions of:

- $\underline{X}$: (X,Y) is a gaussian vector $\Rightarrow X \sim \mathcal{N}(\, , \,)$.

  Furthermore, $(X,Y) \sim \mathcal{N}_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \Gamma\right) \Rightarrow X \sim \mathcal{N}(0,1)$.

- $\underline{Y - a_0 X} = Y - \gamma X$: $\left.\begin{matrix} Y \sim \mathcal{N}(0,1) \\ X \sim \mathcal{N}(0,1)\end{matrix}\right\} \Rightarrow Y - \gamma X \sim \mathcal{N}(\, , \,)$

  And $\begin{cases} \mathbb{E}[Y-\gamma X] = \mathbb{E}[Y] - \gamma \mathbb{E}[X] = 0 \\ Var(Y-\gamma X) = Var(Y) + \gamma^2 Var(X) - 2\gamma Cov(X,Y) = 1 + \gamma^2 - 2\gamma^2 = 1-\gamma^2 \end{cases}$

  $= (1-\gamma)(1+\gamma)$

- $(X, Y - a_0 X) = (X, Y - \gamma X)$: We know that (X,Y) is a gaussian vector.

And $\begin{pmatrix} X \\ Y-\gamma X\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\gamma & 1\end{pmatrix}\begin{pmatrix} X \\ Y\end{pmatrix}$ with $A = \begin{pmatrix} 1 & 0 \\ -\gamma & 1\end{pmatrix}$; from proposition 27, $\begin{pmatrix} X \\ Y-\gamma X\end{pmatrix}$ is a gaussian vector.

Hence $(X, Y-\gamma X) \sim \mathcal{N}_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & 0 \\ 0 & 1-\gamma^2\end{pmatrix}\right)$ since $Cov(X, Y-\gamma X) = Cov(X,Y) - \gamma Cov(X,X)$

$= Cov(X,Y) - \gamma Var(X)$

$= \gamma - \gamma = 0$

$\Rightarrow X \perp\!\!\!\perp Y - \gamma X$.

(4) Compute $\mathbb{E}[Y|X]$: we know that $(X,Y) \sim \mathcal{N}_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \Gamma = \begin{pmatrix}1 & \gamma \\ \gamma & 1\end{pmatrix}\right)$

$\Rightarrow (X,Y)$ has a density function.

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{+\infty} y \, f_{Y|X=x}(y) \, dy \quad \text{with } f_{Y|X=x}(y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)}$$

Then $$f_{Y|X=x}(y) = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^2 \frac{1}{\sqrt{det(\Gamma)}} e^{-\frac{1}{2}(z-\mu)^T \Gamma^{-1}(z-\mu)}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}$$ with $\left(z = \begin{pmatrix}x\\y\end{pmatrix}\right.$ and $\left.\mu = \begin{pmatrix}0\\0\end{pmatrix}\right)$ with $det(\Gamma) > 0$ if

$\gamma \in \,]-1;1[$.

First compute Γ^{-1}: $\Gamma^{-1} = \frac{1}{\det(\Gamma)} {}^t\begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} = \frac{1}{1-r^2}\begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}$ 12

Hence, $\Gamma^{-1} = \begin{pmatrix} \frac{1}{1-r^2} & -\frac{r}{1-r^2} \\ -\frac{r}{1-r^2} & \frac{1}{1-r^2} \end{pmatrix}$. We now study ${}^t(x-0, y-0)\,\Gamma^{-1}\begin{pmatrix} x-0 \\ y-0 \end{pmatrix}$:

Then $(x-0; y-0)\begin{pmatrix} \frac{1}{1-r^2} & -\frac{r}{1-r^2} \\ -\frac{r}{1-r^2} & \frac{1}{1-r^2} \end{pmatrix}\begin{pmatrix} x-0 \\ y-0 \end{pmatrix} = \left(\frac{x}{1-r^2} - \frac{ry}{1-r^2}, \frac{-rx}{1-r^2} + \frac{y}{1-r^2}\right)\begin{pmatrix} x \\ y \end{pmatrix}$

$$= \frac{x^2}{1-r^2} - \frac{rxy}{1-r^2} - \frac{rxy}{1-r^2} + \frac{y^2}{1-r^2} = \frac{x^2 - 2rxy + y^2}{1-r^2} = \frac{(rx-y)^2 + x^2(1-r^2)}{1-r^2}$$

thus $f_{Y|X=x}(y) = \frac{\cancel{\sqrt{2\pi}}}{e^{-\frac{x^2}{2}}}\left(\frac{1}{\sqrt{2\pi}}\right)^{\cancel{2}} \frac{1}{\sqrt{1-r^2}} e^{-\frac{1}{2}\frac{(rx-y)^2 + x^2(1-r^2)}{1-r^2}}$

$$= \frac{1}{\sqrt{2\pi(1-r^2)}} e^{\cancel{\frac{1}{2}x^2} - \frac{1}{2}\frac{(rx-y)^2 + \cancel{x^2(1-r^2)}}{1-r^2}}$$

$$= \frac{1}{\sqrt{2\pi(1-r^2)}} e^{-\frac{1}{2}\frac{(y-rx)^2}{1-r^2}}$$

We realize that this corresponds to the density of a Gaussian distribution!

Therefore $Y|X=x \sim \mathcal{N}(rx, 1-r^2)$

And then $\boxed{\mathbb{E}[Y|X=x] = rx.}$

Exercice 23 Y counts how many times a year a machine is not working. Y is linked to the age X of the machine.

For some given age x, the distribution of Y follows a Poisson law with parameter $\mu_Y(x) = 1 + \ln(x)$. We have:

$$\forall k \in \mathbb{N}, \ \mathbb{P}(Y=k \mid Y=x) = e^{-\mu_Y(x)} \frac{\mu_Y(x)^k}{k!}$$

Here is the distribution of ages for the machine:

x	1	2	3	4
$p_X(x)$	0,1	0,2	0,3	0,4

$\Rightarrow$ $\mathbb{P}(X=1) = 0{,}1$, $\mathbb{P}(X=2) = 0{,}2$, $\mathbb{P}(X=3) = 0{,}3$, $\mathbb{P}(X=4) = 0{,}4$

(1) What does the parameter $\mu_Y(x)$ represent? Comment about its expression.

$\mu_Y(x)$ represents the mean number of breakdown each year, and depends on the age x of the machine. It is thus related to aging, and is naturally increasing with x.

(2) What is the distribution of (X, Y)?

Both Y and X are discrete random variables, meaning that we are interested in quantifying $\forall (k,\ell) \in \mathbb{N} \times \{1;2;3;4\}$, $\mathbb{P}(Y=k, X=\ell)$.

- $X=1$: $\mathbb{P}(Y=k, X=1) = \mathbb{P}(Y=k \mid X=1)\,\mathbb{P}(X=1) = 0{,}1 \ \mathcal{P}(1)$
- $X=2$: $\mathbb{P}(Y=k, X=2) = 0{,}2 \ \mathcal{P}(1+\ln(2))$
- $X=3$: $\mathbb{P}(Y=k, X=3) = 0{,}3 \ \mathcal{P}(1+\ln(3))$
- $X=4$: $\mathbb{P}(Y=k, X=4) = 0{,}4 \ \mathcal{P}(1+\ln(4))$.

(3) I buy a second-hand machine, and I do not know about its age. What is the distribution of annual breakdowns?

A: We have to consider all the possibilities for the age of the machine. We thus get:

$$\mathbb{P}(Y=k) = \sum_{\ell=1}^{4} \mathbb{P}(Y=k, X=\ell) = \sum_{\ell=1}^{4} \mathbb{P}(Y=k \mid X=\ell)\, \mathbb{P}(X=\ell)$$

Thus $F_Y(k) = 0{,}1\, F_{\mathcal{P}(1)} + 0{,}2\, F_{\mathcal{P}}[1+\ln(2)] + 0{,}3\, F_{\mathcal{P}}(1+\ln(3)) + 0{,}4\, F_{\mathcal{P}}(1+\ln(4))$

This is indeed a discrete mixture of Poisson distributions, with 4 components.

Exercices 25-26 See previous chapter, about inequalities.

Exercice 27 Let X be a random variable with a Cauchy distribution, and $(X_n)_{n\geqslant 0}$ a sequence of iid r.v. with same distribution as X.

Q: Denote by S_n the sum $S_n = \sum_{i=1}^{n} X_i$. Show that $\frac{S_n}{n}$ converges towards some given distribution.

A: Remind that here $X \sim \text{Cauchy}(x_0, a)$ with $\begin{cases} x_0 \in \mathbb{R} \text{ (location param)} \\ a > 0 \text{ (scale param)} \end{cases}$

X takes values in $\mathbb{R}$.

X has the following density function: $f_X(x;(x_0,a)) = \dfrac{1}{\pi a\left[1+\left(\frac{x-x_0}{a}\right)^2\right]}$

The standard Cauchy distribution considers $\begin{cases} x_0 = 0 \\ a = 1 \end{cases}$

The characteristic function is given by:

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \int_{-\infty}^{+\infty} f_X(x;(x_0,a))\, e^{itx}\, dx = e^{itx_0 - a|t|}$$

Since the Cauchy distribution does not have a finite second-order moment, it is impossible to use asymptotic results like the CLT.

Let us use then the characteristic functions: we know that $(X_i)_{i=1,\dots,n}$ is a sequence of iid r.v., they are thus $\perp\!\!\!\perp$ and we have then

$$\varphi_{X_1+\dots+X_n}(t) = \varphi_{X_1}(t)\dots\varphi_{X_n}(t) = \left(e^{itx_0 - a|t|}\right)^n = e^{intx_0 - na|t|}$$

The Cauchy distribution also has the property that if $X_i \sim \text{Cauchy}(x_0, a)$ then $k X_i$ has the characteristic function $\varphi_{kX_i}(t) = \varphi_{X_i}(kt)$.

It follows that $\frac{1}{n}\sum_{i=1}^n X_i$ has the characteristic function $e^{\left(\frac{1}{n}(e^{itx_0 - a|t|})\right)^n}$

$= e^{itx_0 - a|t|}$; and because the characteristic function identifies the distribution, we have $\frac{1}{n}\sum_{i=1}^n X_i \overset{\mathcal{L}}{\sim} \text{Cauchy}(x_0, a)$. (not asymptotic, this is an exact result).

Exercise 28 Let $(X_n)_{n\geq 0}$ be a sequence of iid r.v. with density

$$f(x) = \frac{3}{4}(1-x^2)\mathbb{1}_{|x|\leq 1}, \quad \text{and introduce } \xi_n = \max(X_1, \dots, X_n).$$

(1) Show that $\xi_n \xrightarrow[n\to+\infty]{\mathbb{P}} 1 \iff \forall \varepsilon > 0, \lim_{n\to+\infty} \mathbb{P}(|\xi_n - 1| > \varepsilon) = 0$.

$$\mathbb{P}(|\underbrace{\max(X_1,\dots,X_n)}_{\leq 1} - 1| > \varepsilon) = \mathbb{P}(1 - \max(X_1,\dots,X_n) > \varepsilon)$$

$$= \mathbb{P}(\max(X_1,\dots,X_n) < 1-\varepsilon) = \mathbb{P}(X_1 < 1-\varepsilon, \dots, X_n < 1-\varepsilon)$$

$$\overset{X_i \perp\!\!\!\perp}{=} (\underbrace{\mathbb{P}(X_i < 1-\varepsilon)}_{\in]0,1[})^n \xrightarrow[n\to+\infty]{} 0.$$

$\in]0,1[$ → see the density function

(2) Prove that $\sqrt{n}(1-\xi_n)$ converges in distribution, and give the density of this limiting distribution.

(14)

Maybe use the theorem by Paul Lévy; or results about convergence in distribution and convergence of characteristic functions.

$$\mathbb{P}\left(\sqrt{n}(1-\xi_n) \le t\right) = \mathbb{P}\left(1-\xi_n \le \frac{t}{\sqrt{n}}\right) = \mathbb{P}\left(\xi_n \geq 1-\frac{t}{\sqrt{n}}\right)$$

$$= \mathbb{P}\left(\max(X_1,\dots,X_n) \geq 1-\frac{t}{\sqrt{n}}\right) = 1 - \mathbb{P}\left(\max(X_1,\dots,X_n) < 1-\frac{t}{\sqrt{n}}\right)$$

$$\overset{X_i \text{ iid}}{=} 1 - \left(\mathbb{P}\left(X_i < 1-\frac{t}{\sqrt{n}}\right)\right)^n, \text{ with}$$

$$\mathbb{P}\left(X_i \geq 1-\frac{t}{\sqrt{n}}\right) = \begin{cases} 0 & \text{if } t<0 \\ \displaystyle\int_{1-\frac{t}{\sqrt{n}}}^{1} f_X(x)\,dx & \text{if } t \geq 0. \end{cases}$$

Hence $$\int_{1-\frac{t}{\sqrt{n}}}^{1} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{1-\frac{t}{\sqrt{n}}}^{1}$$

$$= \frac{3}{4}\left[\cancel{1} - \frac{1}{3} - \cancel{\left(1-\frac{t}{\sqrt{n}}\right)} + \frac{\left(1-\frac{t}{\sqrt{n}}\right)^3}{3}\right] = \frac{3}{4}\left[-\frac{1}{3} + \frac{t}{\sqrt{n}} + \frac{1}{3}\left(1-\frac{t}{\sqrt{n}}\right)^3\right]$$

$$= \frac{3}{4}\left[\frac{t}{\sqrt{n}} - \frac{1}{3}\left(1-\left(1-\frac{t}{\sqrt{n}}\right)^3\right)\right] \simeq \frac{3}{4}\frac{t^2}{n} \text{ when } \begin{cases} t \geq 0 \\ n \to +\infty \end{cases}$$

Exercice 31 Let $X_1, \dots, X_{1000}$ be Uniform random variables on $[0,1]$.

We denote by M how many times the realizations of those r.v. belongs to $[\frac{1}{4};\frac{3}{4}]$.

Q: Using the Gaussian approximation, determine $\mathbb{P}(|M-500|>20)$.

A: Let N_i be a Bernoulli r.v. such that $N_i = \begin{cases} 1 & \text{if } x_i \in [\frac{1}{4};\frac{3}{4}] \\ 0 & \text{otherwise} \end{cases}$

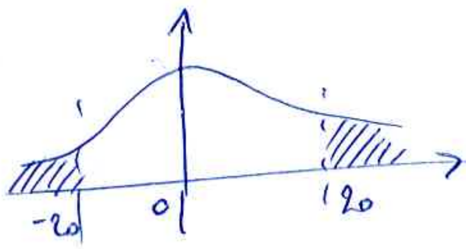
We have $M = \sum_{i=1}^{1000} N_i$, with $N_i \sim \mathcal{B}(\frac{1}{2})$

N_i are iid r.v., with $\begin{cases} \mathbb{E}[N_i] = \frac{1}{2} < \infty \\ Var(N_i) = 1/4 < \infty \end{cases}$, we can therefore apply the CLT:

$$M = \sum_{i=1}^{1000} N_i \underset{n\to+\infty}{\sim} \mathcal{N}\Big(\underbrace{n}_{=1000} \mathbb{E}[N_i], \underbrace{n}_{=1000} Var(N_i)\Big) = \mathcal{N}(500, 250)$$

Then we get $M-500 \sim \mathcal{N}(0,250)$

And thus $Z = \frac{M-500}{\sqrt{250}} \sim \mathcal{N}(0,1)$



Finally, $\mathbb{P}(|M-500|>20) = \mathbb{P}\left(|Z| > \frac{20}{\sqrt{250}}\right) = \mathbb{P}(|Z| > 1{,}265)$

$= 1 - \mathbb{P}(-1{,}265 \leq Z \leq 1{,}265)$

$= 1 - \big(F_Z(1{,}265) - F_Z(-1{,}265)\big) = 1 - F_Z(1{,}265) + F_Z(-1{,}265)$.

$= 1 - 0{,}897 + 0{,}103 = 0{,}206$.