M1 MAS
Statistics

TD Chapter 1

Exercise 1 Q: Let $(A_i)_{i\geq 0}$ a sequence of events such that $\mathbb{P}(A_i) = 1$ for all $i \geq 0$. Prove that $\mathbb{P}(\cap A_i) = 1$.

A: We prove this by induction.

- Initialization step: $i = 0$: the sequence is only the event A_0, such that $\mathbb{P}(A_0) = 1$.

$i = 1$: we have $\mathbb{P}(A_0) = \mathbb{P}(A_1) = 1$.

Then, $\underbrace{\mathbb{P}(A_0 \cap A_1)}_{\in [0,1]} = \underbrace{\mathbb{P}(A_0)}_{=1} + \underbrace{\mathbb{P}(A_1)}_{=1} - \underbrace{\mathbb{P}(A_0 \cup A_1)}_{\in [0,1]}$ since probability.

Therefore $\mathbb{P}(A_0 \cup A_1) = 1$ and thus $\mathbb{P}(A_0 \cap A_1) = 1$.

- Induction step: we assume that the property at rank i.

Then we have $\begin{cases} \mathbb{P}(A_0) = 1 \\ \mathbb{P}(A_1) = 1 \\ \vdots \\ \mathbb{P}(A_i) = 1 \\ \mathbb{P}(A_{i+1}) = 1 \end{cases}$, with $\mathbb{P}(\cap_i A_i) = 1$.

We study $\mathbb{P}(\cap_{i+1} A_i) = \mathbb{P}(A_0 \cap A_1 \cap \dots \cap A_{i+1})$

$$= \mathbb{P}(\underbrace{A_0 \cap A_1 \cap \dots \cap A_i}_{B} \cap A_{i+1}) = \mathbb{P}(B \cap A_{i+1})$$

Therefore, $\underbrace{\mathbb{P}(\cap_{i+1} A_i)}_{\in [0,1]} = \mathbb{P}(B) + \mathbb{P}(A_{i+1}) - \mathbb{P}(B \cup A_{i+1})$

$$= \underbrace{\mathbb{P}(\cap_i A_i)}_{=1} + \underbrace{\mathbb{P}(A_{i+1})}_{=1} - \underbrace{\mathbb{P}(\cup_{i+1} A_i)}_{\in [0,1]}$$

This way, $\mathbb{P}(\bigcup_{i \geq 1} A_i) = 1$, and thus $\mathbb{P}(\bigcap_{i \geq 1} A_i) = 1$.

Exercice 2

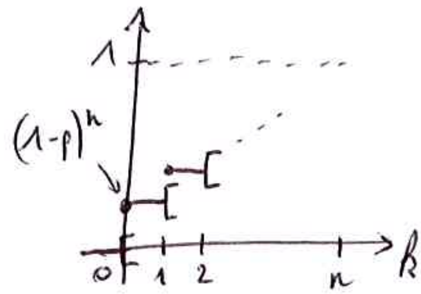
Course

Exercice 3

(a) $X \sim \mathcal{B}(n,p) \Rightarrow \mathbb{P}(X=k) = C_n^k \, p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$ with values in $\{0;1;2;\dots;n\}$

$$F_X(x) = \mathbb{P}(X \leq x) = \sum_{k=0}^{\lfloor x \rfloor} \mathbb{P}(X=k)$$

$$= \sum_{k=0}^{\lfloor x \rfloor} C_n^k \, p^k (1-p)^{n-k}$$



Ainsi $F_X(x) = \begin{cases} 0 & \text{si } x < 0 \\ \sum_{k=0}^{\lfloor x \rfloor} C_n^k \, p^k (1-p)^{n-k} & \text{si } 0 \leq x < n \\ 1 & \text{si } x \geq n \end{cases}$

(b) $X \sim \mathcal{P}(\lambda) \Rightarrow \mathbb{P}(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$, with values in $\mathbb{N}$.

Ainsi $F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sum_{k=0}^{\lfloor x \rfloor} e^{-\lambda} \frac{\lambda^k}{k!} & \text{otherwise} \end{cases}$

(c) Loi exponentielle : $X \sim \mathcal{E}xp(\lambda) \Rightarrow f_X(x) = \lambda e^{-\lambda x} \mathbb{1}_{\{x > 0\}}$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{-\infty}^{x} \lambda e^{-\lambda u} \mathbb{1}_{\{u > 0\}}\,du = \int_0^x \lambda e^{-\lambda u}\,du$$

$$= \cancel{\lambda} \left[-\frac{1}{\cancel{\lambda}} e^{-\lambda u} \right]_0^x = -e^{-\lambda x} + 1$$

donc $F_X(x) = (1 - e^{-\lambda x}) \mathbb{1}_{\{x > 0\}}$.

(d) The distribution with density function given by

$$f(x) = \frac{1}{2} x e^{-x^2} \mathbb{1}_{\{x \geq 0\}}$$

Thus $F_X(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{x} \frac{1}{2} u e^{-u^2} \mathbb{1}_{\{u \geq 0\}}\,du$

$$= -\frac{1}{4} \int_0^x -2u e^{-u^2}\,du = -\frac{1}{4}\left[e^{-u^2}\right]_0^x = -\frac{1}{4}\left(e^{-x^2} - 1\right)$$

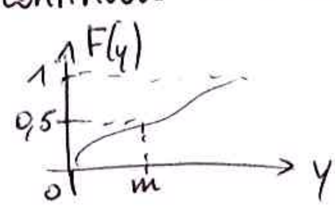
then $F_X(x) = \frac{1}{4}\left(1 - e^{-x^2}\right) \mathbb{1}_{\{x \geq 0\}}$.

Exercise 4 Consider a distribution with cdf F. The number m is the median of F if $\lim_{y \to m^-} F(y) \leq \frac{1}{2} \leq F(m)$.

Q: Does "m" always exist? Is "m" unique?

A: There are 3 distinct cases:

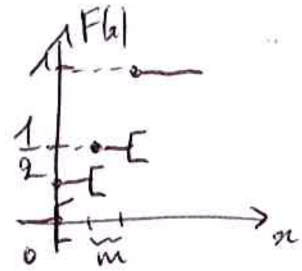
- continuous distribution: F is strictly $\nearrow$, $F \in [0,1]$, F is continuous $\Rightarrow$ From the theorem of intermediary values, there exists a unique point such that $F(m) = \frac{1}{2}$.



In this case $\lim_{y \to m^-} F(y) = F(m) = \frac{1}{2}$.

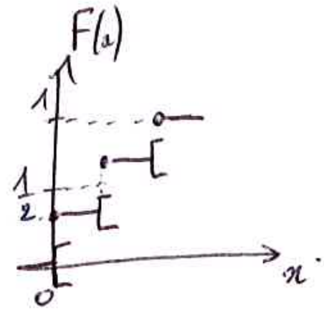
- discrete distribution: F is increasing, $F \in [0,1]$, F is càdlàg $\Rightarrow$ then there exist 2 different situations...

- if $F(x) = \frac{1}{2}$ is located on a plateau: "m" is not unique, and can equal any point of the plateau



- if $F(x) = \frac{1}{2}$ is located where there is a jump, then the median equals the limit on the left since the value on the right would lead to have $F(m) > \frac{1}{2}$.

  In this case, $\lim_{y \to m^-} F(y) = \frac{1}{2} < F(m)$



Exercice 5 Let X be a real-valued random variable, with cdf F.

For all $(a,b) \in \mathbb{R}^2$, $a < b$. We have:

$$\begin{cases} \mathbb{P}(X \in]a,b]) = F(b) - F(a) \\ \mathbb{P}(X \in]a,b[) = F(b^-) - F(a), \text{ where } F(b^-) = \lim_{x \to b^-} F(x) \\ \mathbb{P}(X \in [a,b]) = F(b) - F(a^-) \\ \mathbb{P}(X \in [a,b[) = F(b^-) - F(a^-) \\ \mathbb{P}(X = a) = 0 \end{cases}$$

Exercice 6 Let X be a real-valued random variable with cdf F.

F is continuous, and G is a continuous strictly increasing function over $\mathbb{R}$. Find out the cdf of the following random variables:

- $F_{-X}(x) = \mathbb{P}(-X \leq x) = \mathbb{P}(X \geq -x) = 1 - \mathbb{P}(X < -x) = 1 - F(-x)$.

- $F_{X^2}(x) = \mathbb{P}(X^2 \leq x) = \mathbb{P}(-\sqrt{x} \leq X \leq \sqrt{x}) = \begin{cases} F(\sqrt{x}) - F(-\sqrt{x}) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

- $F_{|X|}(x) = \mathbb{P}(|X| \leq x) = \mathbb{P}(-x \leq X \leq x) = \begin{cases} F(x) - F(-x) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

- $F_{\sin(X)}(x) = \mathbb{P}(\sin(X) \leq x) \underset{\text{arcsin } \nearrow}{=} \mathbb{P}(X \leq \arcsin(x)) = F(\arcsin(x))$.

- $F_{X^+}(x) = \mathbb{P}(\max(0,X) \leq x) = \mathbb{P}(0 \leq x, X \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ F(x) & \text{if } x \geq 0 \end{cases}$

- $F_{X^-}(x) = \mathbb{P}(\max(0,-X) \leq x) = \mathbb{P}(-\min(0,X) \leq x)$

  $= \mathbb{P}(\min(0,X) \geq -x) = \mathbb{P}(0 \geq -x, X \geq -x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - F(-x) & \text{if } x \geq 0 \end{cases}$

- $F_{G^{-1}(X)}(x) = \mathbb{P}(G^{-1}(X) \leq x) = \mathbb{P}(X \leq G(x)) = F(G(x))$

- $F_{F(X)}(x) = \mathbb{P}(F(X) \leq x) = \mathbb{P}(F^{-1}(F(X)) \leq F^{-1}(x)) = \mathbb{P}(X \leq F^{-1}(x))$

  $= F(F^{-1}(x)) = x \Rightarrow F(X)$ follows a Uniform distribution!

- $F_{G^{-1}(F(X))}(x) = \mathbb{P}(G^{-1}(F(X)) \leq x) = \mathbb{P}(F(X) \leq G(x)) = \mathbb{P}(X \leq F^{-1}(G(x)))$

  $= F(F^{-1}(G(x))) = G(x)$

Exercise 7 Let X be a real-valued random variable with cdf F, and $(a,b) \in \mathbb{R}^2$ such that $a < b$.

- What is the cdf of $Y = \begin{cases} a & \text{if } X < a \\ X & \text{if } a \leq X \leq b \\ b & \text{else} \end{cases}$

$\mathbb{P}(Y \leq x) = \mathbb{P}(Y \leq x, X < a) + \mathbb{P}(Y \leq x, a \leq X \leq b) + \mathbb{P}(Y \leq x, X > b)$

$= \mathbb{P}(Y \leq x \mid X < a)\,\mathbb{P}(X < a) + \mathbb{P}(Y \leq x \mid a \leq X \leq b)\,\mathbb{P}(a \leq X \leq b) + \mathbb{P}(Y \leq x \mid X > b)\,\mathbb{P}(X > b)$

Given the definition of Y, Y takes values in $\{a; b; X\}$, and values of X when $a \leq X \leq b$.

Then, if we are interested in $\mathbb{P}(Y \leq y) = F_Y(y)$, it is clear that it depends on the position of y with respect to a and b.

- $y < a$: $\mathbb{P}(Y \leq y) = 0$, obviously by definition Y never takes values strictly lower than a.
- $y = a$: $\mathbb{P}(Y = a) = \mathbb{P}(X < a) = F_X(a)$.
- $a < y < b$: by definition of Y,
  $$\mathbb{P}(Y \leq y) = \mathbb{P}(Y \leq y, a \leq X \leq b) = \mathbb{P}(X \leq y, a \leq X \leq b) = F_X(y)$$
  $\Big\}$ • $a \leq y < b$: $\mathbb{P}(Y \leq y) = F_X(y)$
- $y = b$: $\mathbb{P}(Y = b) = \mathbb{P}(X > b) = 1 - \mathbb{P}(X \leq b) = 1 - F_X(b)$
- $y \geq b$: $\mathbb{P}(Y \leq y) = 1$ by definition since b is the largest value that Y can reach.
  $$= \mathbb{P}(X < a) + \mathbb{P}(a \leq X \leq b) + \mathbb{P}(X > b)$$
  $$= \mathbb{P}(\{X < a\} \cup \{a \leq X \leq b\} \cup \{X > b\}) = \mathbb{P}(X \in]-\infty; +\infty[) = 1.$$

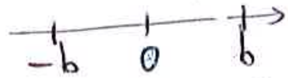
We can conclude:
$$F_Y(y) = \begin{cases} 0 & \text{if } y < a \\ F_X(y) & \text{if } a \leq y < b \\ 1 & \text{if } y \geq b \end{cases}$$

$\rightarrow$ F_Y is increasing.

- What is the cdf of $Z = \begin{cases} X & \text{if } |X| \leq b \\ 0 & \text{else} \end{cases}$ $\Rightarrow$ Z takes values $\{0\}$ and the values of X if $|X| \leq b$.

$$\mathbb{P}(Z \leq x) = \mathbb{P}(Z \leq x, |X| \leq b) + \mathbb{P}(Z \leq x, |X| > b)$$
$$= \mathbb{P}(Z \leq x \mid |X| \leq b)\, \mathbb{P}(|X| \leq b) + \mathbb{P}(Z \leq x \mid |X| > b)\, \mathbb{P}(|X| > b)$$
$$= \mathbb{P}(Z \leq x \mid -b \leq X \leq b)\, \mathbb{P}(-b \leq X \leq b) + \mathbb{P}(Z \leq x \mid |X| > b)\, \mathbb{P}(|X| > b)$$
$$= \mathbb{P}(X \leq x \mid -b \leq X \leq b) + \dots$$
$\times\, \mathbb{P}(-b \leq X \leq b)$

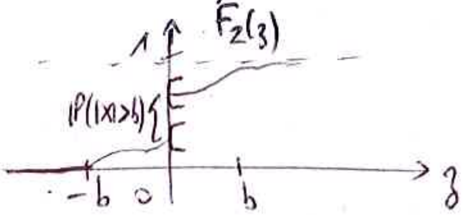
not independent...

Once again, it depends on the value of z: 

- $z < -b$: $\mathbb{P}(Z \leq z) = 0$ since by definition Z cannot take values lower than $-b$.
- $z > b$: $\mathbb{P}(Z \leq z) = 1$ since by definition Z cannot take values greater than b.
- $-b \leq z \leq b$: $\mathbb{P}(Z \leq z) = \underbrace{\mathbb{P}(Z \leq z, \mathbb{1}_{X \neq Z})}_{①} + \underbrace{\mathbb{P}(Z \leq z, \mathbb{1}_{X = Z})}_{②}$

$$① = \begin{cases} 0 & \text{if } z < 0 \text{ and } z \geq -b \\ \mathbb{P}(Z = 0) & \text{if } z \geq 0 \text{ and } z \leq b \end{cases} = \mathbb{P}(|X| > b) = F_X(-b) + 1 - F_X(b)$$

$$② = \mathbb{P}(-b \leq X \leq z) \text{ if } -b \leq z \leq b = F_X(z) - F_X(-b)$$



Therefore $\mathbb{P}(Z \leq z) = \begin{cases} 0 & \text{if } z < -b \\ F_X(z) - F_X(-b) & \text{if } -b \leq z < 0 \\ F_X(z) - \cancel{F_X(-b)} + \underbrace{\cancel{F_X(-b)} + 1 - F_X(b)}_{\mathbb{P}(Z=0)} = 1 - F_X(b) + F_X(z) & \text{if } 0 \leq z < b \\ 1 & \text{if } z \geq b. \end{cases}$

Exercise 8 Find the value of c such that the following function f is a density function.

(a) $f(x) = \begin{cases} c\,x^{-d} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$

- f is continuous
- f is positive that $c > 0$

- $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} c\,x^{-d}\,\mathbb{1}_{\{x > 1\}}\,dx = \int_{1}^{+\infty} c\,x^{-d}\,dx$

$$= c\left[+\frac{x^{-d+1}}{-d+1}\right]_1^{+\infty} \overset{d>1}{=} c\left[0 - \frac{1}{-d+1}\right] = \frac{-c}{1-d}$$

this integral must sum to 1 $\Rightarrow \dfrac{-c}{1-d} = 1 \Rightarrow \boxed{c = -1 + d}$

(b) $f(x) = c e^{x}(1+e^{x})^{-2}$ is $\begin{cases} \text{continuous} \\ \text{positive provided that } c>0 \end{cases}$

- $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} c e^{x}(1+e^{x})^{-2}\,dx = 1 \Leftrightarrow -c\left[(1+e^{x})^{-1}\right]_{-\infty}^{+\infty} = 1$

$\Leftrightarrow -c(0-1) = 1 \Rightarrow \boxed{c=1}$.

Exercise 9 Let a and p such that $\begin{cases} a>0 \\ 0<p<1 \end{cases}$.

Let X be a random variable following a Poisson distribution $\mathcal{P}(a)$.
Let Y be an integer-valued random variable such that, $\forall\, 0 \leq k \leq n$,

$$\mathbb{P}(Y=k \mid X=n) = C_n^k\, p^k (1-p)^{n-k}.$$

- What is the distribution of Y?

We first notice that $Y|X \sim \mathcal{B}(X,p)$.

$$\mathbb{P}(Y=k) = \sum_{i=0}^{+\infty} \mathbb{P}(Y=k \mid X=i)\,\mathbb{P}(X=i) = \sum_{i=0}^{+\infty} C_i^k\, p^k (1-p)^{i-k} e^{-a}\frac{a^i}{i!}$$

$$= \sum_{i=0}^{+\infty} \frac{\cancel{i!}}{k!(i-k)!} p^k (1-p)^{i-k} e^{-a}\frac{a^i}{\cancel{i!}} = e^{-a}\frac{p^k}{k!}\sum_{i=0}^{+\infty}\frac{(1-p)^{i-k}}{(i-k)!}a^{i-k+k}$$

$$= \frac{p^k}{k!}e^{-a}a^k \sum_{i=0}^{+\infty}\frac{(a(1-p))^{i-k}}{(i-k)!} = \frac{(ap)^k}{k!}e^{-a} \times e^{a(1-p)}$$

$$= \frac{(ap)^k}{k!}e^{\cancel{-a}\cancel{+a}-ap} = e^{-ap}\frac{(ap)^k}{k!}$$

Thus $Y \sim \mathcal{P}(ap)$.

- $\mathbb{P}(Z=k) = \mathbb{P}(X-Y=k)$

$$= \mathbb{P}(Y = X-k)$$

$$= \sum_{n=0}^{+\infty} \mathbb{P}(Y = X-k \mid X=n)\, \mathbb{P}(X=n)$$

$$= \sum_{n=0}^{+\infty} \mathbb{P}(Y=n-k \mid X=n)\, \mathbb{P}(X=n) = \sum_{n=0}^{+\infty} C_n^{n-k} p^{n-k} (1-p)^{n-(n-k)} e^{-a} \frac{a^n}{n!}$$

$$= \sum_{n=0}^{+\infty} \frac{\cancel{n!}}{(n-k)!\,(n-(n-k))!} p^{n-k} (1-p)^k e^{-a} \frac{a^n}{\cancel{n!}} = \sum_{n=0}^{+\infty} \frac{1}{(n-k)!\,k!} p^{n-k} (1-p)^k e^{-a} a^n$$

$$= \frac{(1-p)^k}{k!} e^{-a} \sum_{n\geq 0} \frac{1}{(n-k)!} p^{n-k} a^{n-k+k}$$

$$= \frac{(1-p)^k}{k!} e^{-a} a^k \sum_{n=0}^{+\infty} \frac{(ap)^{n-k}}{(n-k)!} = \frac{(1-p)^k}{k!} e^{-a} a^k e^{ap}$$

$$= e^{-a+ap} \frac{(a(1-p))^k}{k!} = e^{-a(1-p)} \frac{(a(1-p))^k}{k!}, \text{ so } Z \sim \mathcal{P}(a(1-p)).$$

Exercice 10 Let T be an integer-valued random variable, such that

- $\forall n \in \mathbb{N}$, $\mathbb{P}(T \geq n) > 0$
- $\forall n,m \in \mathbb{N}$, $\mathbb{P}(T \geq n+m \mid T \geq n) = \mathbb{P}(T \geq m)$.

1- T is considered "without memory" since the probability to reach the threshold "$n+m$" given that we reached "n" is the same of the probability to simply reach the threshold "m". This means that what happens in the past has no influence on the future.

2. If $T \sim \mathcal{G}(q)$ then $\mathbb{P}(T=n) = (1-q)^n q$ (success at the $(n+1)$-th experiment).

- $\mathbb{P}(T \geq m) = \sum_{k=m}^{+\infty} \mathbb{P}(T=k) = \sum_{k=m}^{+\infty} (1-q)^k q = q \sum_{k=m}^{+\infty} (1-q)^k$

$$= q\left[\sum_{k=0}^{+\infty} (1-q)^k - \sum_{k=0}^{m-1} (1-q)^k\right] = q\left[\frac{1}{1-(1-q)} - \frac{1-(1-q)^m}{1-(1-q)}\right]$$

$$= \cancel{q}\left[\frac{1}{\cancel{q}} - \frac{1-(1-q)^m}{\cancel{q}}\right] = \cancel{1} - \cancel{1} + (1-q)^m = (1-q)^m$$

(easy since that corresponds to one single possible event: $\underbrace{\text{échec} \cap \text{échec} \cap \dots \cap \text{échec}}_{m \text{ fois}} \cap$ succès

- $\mathbb{P}(T \geq n+m \mid T \geq n) = \frac{\mathbb{P}(T \geq n+m, T \geq n)}{\mathbb{P}(T \geq n)} = \frac{\mathbb{P}(T \geq n+m)}{\mathbb{P}(T \geq n)}$

$$= \frac{(1-q)^{n+m}}{(1-q)^n} = (1-q)^{n+m-n} = (1-q)^m = \mathbb{P}(T \geq m).$$

Exercice 11 Similar to exercice 10.

Exercice 12 Let X be a random variable that takes values in $\{3; -1; 1\}$, and Y which takes values in $\{-1; 3\}$.

- $X = \sum_i \alpha_i \mathbb{1}_{A_i} = 3\mathbb{1}_{\{X=3\}} - \mathbb{1}_{\{X=-1\}} + \mathbb{1}_{\{X=1\}}$
- $Y = 3\mathbb{1}_{\{Y=3\}} - \mathbb{1}_{\{Y=-1\}}$
- $e^X = e^3 \mathbb{1}_{\{X=3\}} + e^{-1} \mathbb{1}_{\{X=-1\}} + e \mathbb{1}_{\{X=1\}}$
- $X^2 = 9\mathbb{1}_{\{X=3\}} + \mathbb{1}_{\{X=-1 \cup X=1\}}$
- $X+Y = -2\mathbb{1}_{\{X=-1, Y=-1\}} + 6\mathbb{1}_{\{X=3, Y=3\}} + \dots$

Exercice 13

We throw a die, and $\begin{cases} \text{win } 2€ \text{ in case of prime number} \\ \text{loose } 2€ \text{ in case of even number} \end{cases}$ (6)

Denote by X the gain.

1. result of the die $\nearrow$

$L=1 \Rightarrow X=2$

$L=2 \Rightarrow X=2-2=0$

$L=3 \Rightarrow X=2$

$L=4 \Rightarrow X=-2$

$L=5 \Rightarrow X=2$

$L=6 \Rightarrow X=-2$

2. $X = 2\,\mathbb{1}_{\{L=1 \cup L=3 \cup L=5\}} - 2\,\mathbb{1}_{\{L=4 \cup L=6\}}$

3. • $\mathbb{P}(X=0) = 1/6$

$\mathbb{P}(X=2) = 1/2$

$\mathbb{P}(X=-2) = 1/3$

• $\mathbb{E}[X] = 0 \times \frac{1}{6} + 2 \times \frac{1}{2} + (-2) \times \frac{1}{3} = 1 - \frac{2}{3} = \frac{1}{3}$

Exercice 14

Denote by X a random variable.

(1) $X \sim \mathcal{B}(n,p)$: $X \in \{0;1;\dots;n\}$

The first term ($k=0$) equals 0 $\Rightarrow$ we can remove it from the sum.

$$\mathbb{E}[X] = \sum_{k=0}^{n} k\,\mathbb{P}(X=k) = \sum_{k=0}^{n} k\, C_n^k\, p^k (1-p)^{n-k} = \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= n \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} p^k (1-p)^{n-k} = n \sum_{k=1}^{n} C_{n-1}^{k-1} p^k (1-p)^{n-k}$$

$$= n \sum_{k=0}^{n} C_{n-1}^{k} p^{k+1} (1-p)^{n-(k+1)} = np \sum_{k=0}^{n} C_{n-1}^{k} p^{k} (1-p)^{n-k-1}$$

$$= np \underbrace{\sum_{k=0}^{n} C_{n-1}^{k} p^{k} (1-p)^{n-1-k}}_{\text{binôme de Newton}} = np\,(\cancel{p} + (1-\cancel{p}))^{n-1} = np$$

Or we could also use : $\begin{cases} X = \sum_{i=1}^{n} Y_i \\ Y_i \sim \mathcal{B}(p) \\ Y_i \perp\!\!\!\perp \end{cases} \Rightarrow \mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} Y_i] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = n\,\mathbb{E}[Y_i] = np.$

(2) $X \sim \mathcal{P}(\lambda)$: $X \in \mathbb{N}$.

$$\mathbb{E}[X] = \sum_{k \geq 0} k\, \mathbb{P}(X=k) = \underbrace{\sum_{k \geq 0} k\, e^{-\lambda} \frac{\lambda^k}{k!}}_{1^{st} \text{ term equals } 0} = \sum_{k=1}^{+\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= e^{-\lambda} \sum_{k \geq 1} \frac{\lambda^{k-1+1}}{(k-1)!} = e^{-\lambda} \times \lambda \sum_{k \geq 1} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \underbrace{\sum_{k \geq 0} \frac{\lambda^k}{k!}}_{= e^{\lambda}} = \lambda .$$

(3) $X \sim \mathcal{E}(\lambda)$; X takes values in $\mathbb{R}^{+*} = \mathcal{X}$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx = \int_{-\infty}^{+\infty} x \lambda e^{-\lambda x} \mathbb{1}_{\{x>0\}}\, dx = \int_0^{+\infty} \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x}}_{v'} dx$$

$$= -\int_0^{+\infty} \underbrace{-\lambda e^{-\lambda x}}_{v'} \times \underbrace{x}_{u}\, dx \overset{IPP}{=} -\left(\left[x e^{-\lambda x} \right]_0^{+\infty} - \int_0^{+\infty} e^{-\lambda x} dx \right)$$

$$= -\left(0 - \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_0^{+\infty} \right) = -\left(-\left(0 + \frac{1}{\lambda} \right) \right) = \frac{1}{\lambda} .$$

(4) $X \sim \mathcal{N}(\mu, \sigma^2)$; X takes values in $\mathbb{R}$.

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx = \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx . \text{ Let } g(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Rightarrow g'(x) = -\frac{1}{\cancel{2}\sigma^2} \cancel{2}(x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Hence, $$\mathbb{E}[X] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} (x-\mu+\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \times (-\sigma^2) \int_{-\infty}^{+\infty} -\frac{1}{\sigma^2}(x-\mu+\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\mathbb{E}[X] = \frac{1}{\sqrt{2\pi\sigma^2}}(-\sigma^2)\left[\int_{-\infty}^{+\infty} \underbrace{-\frac{1}{\sigma^2}(x-\mu)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{g'(x)} dx + \mu \int_{-\infty}^{+\infty} -\frac{1}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx\right]$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}(-\sigma^2)\left[\Big[g(x)\Big]_{-\infty}^{+\infty} + \left(-\frac{\mu}{\sigma^2}\right)\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx\right]$$

$$= -\frac{\sigma^2}{\sqrt{2\pi\sigma^2}}\left[\left[e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right]_{-\infty}^{+\infty} - \frac{\mu}{\sigma^2}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx\right]$$

since $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

$$= -\frac{\sigma^2}{\sqrt{2\pi\sigma^2}}(0-0) + \mu \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{f_X(x)} dx = \mu \times 1 = \mu$$

Exercise 15 Let X be an integer-valued random variable.

$$\forall j \in \mathbb{N}, \quad p_j = \mathbb{P}(X=j) \text{ and } q_j = \mathbb{P}(X>j).$$

Prove that $\mathbb{E}[X] = \sum_{j=0}^{+\infty} q_j$.

- In the continuous case, say X takes value in $\mathbb{R}^+$ (it has to be a positive random variable), we have $X = \int_0^{+\infty} \mathbb{1}_{[0,X[}(x)\,dx$

Thus $\mathbb{E}[X] = \int_\Omega X\, d\mathbb{P} = \int_\Omega \left(\int_0^{+\infty} \mathbb{1}_{[0,X[}(x)\,dx\right) d\mathbb{P}$

Fubini-Tonelli since only positive quantities

$$= \int_0^{+\infty} \underbrace{\left(\int_\Omega \mathbb{1}_{[0,X[}(x)\,d\mathbb{P}\right)}_{\mathbb{E}[\mathbb{1}_{\{X>x\}}]} dx = \int_0^{+\infty} \underbrace{\mathbb{P}(X>x)}_{S(x)}\,dx.$$

survival function

- In the same way for the discrete case on integers:
the survival function would be written as an expectation with the formula: $S = \sum_{k=0}^{+\infty} x_k \mathbb{1}_{\{N>k\}}$ - And the same applies then.

Exercise 16 For $\alpha > 0$, we have $\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx$.

The Gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$, denoted by $G(\alpha, \lambda)$, has density function given by:

$$\gamma_{\alpha,\lambda}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} \mathbb{1}_{\mathbb{R}^+}(x).$$

1. Check that $\Gamma(\alpha)$ exists for $\alpha > 0$, and show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.
Then, compute $\Gamma(n)$ for $n \in \mathbb{N}^*$.

- $\Gamma(\alpha+1) = \int_0^{+\infty} e^{-x} x^{\alpha+1-1} dx = \int_0^{+\infty} \underbrace{e^{-x}}_{v'} \underbrace{x^\alpha}_{u} dx \overset{IPP}{=} \left[-e^{-x} x^\alpha\right]_0^{+\infty} - \int_0^{+\infty} -e^{-x} \alpha x^{\alpha-1} dx$

$= 0 + \alpha \int_0^{+\infty} e^{-x} x^{\alpha-1} dx = \alpha\,\Gamma(\alpha).$

- $\Gamma(1) = \int_0^{+\infty} e^{-x} x^0 dx = \int_0^{+\infty} e^{-x} dx = \left[-e^{-x}\right]_0^{+\infty} = 1$

$\Gamma(2) = 1\,\Gamma(1) = 1 \times 1 = 1 = 1!$

$\Gamma(3) = 2\,\Gamma(2) = 2 \times 1 = 2 = 2!$

$\Gamma(4) = 3\,\Gamma(3) = 3 \times 2 \times 1 = 6 = 3!$

$\Gamma(5) = 4\,\Gamma(4) = 4 \times 3 \times 2 \times 1 = 4!$

$\Gamma(n) = (n-1)!$

2- Let X be a $G(\alpha,\lambda)$-distributed random variable.

- $$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x\, \gamma_{\alpha,\lambda}(x)\,dx = \int_{-\infty}^{+\infty} x\, \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} \mathbb{1}_{\mathbb{R}^+}(x)\,dx$$

$$= \int_0^{+\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1+1}\,dx = \int_0^{+\infty} \frac{\lambda^\alpha}{\underbrace{\frac{\Gamma(\alpha+1)}{\alpha}}_{\text{from question 1}}} e^{-\lambda x} x^{\alpha+1-1}\,dx$$

$$= \alpha \int_0^{+\infty} \frac{\lambda^{\alpha+1-1}}{\Gamma(\alpha+1)} e^{-\lambda x} x^{\alpha+1-1}\,dx = \underbrace{\frac{\alpha}{\lambda} \int_0 \underbrace{\frac{\lambda^{\alpha'-1}}{\Gamma(\alpha')} e^{-\lambda x} x^{\alpha'-1}}_{\text{density of } G(\alpha',\lambda)}\,dx}_{=1}$$

$$= \frac{\alpha}{\lambda}.$$

- $$\mathrm{Var}(X) = ? = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 \gamma_{\alpha,\lambda}(x)\,dx = \int_{-\infty}^{+\infty} x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} \mathbb{1}_{\mathbb{R}^+}(x)\,dx$$

$=$

3- Let Y be a random variable such that $Y \sim \mathcal{N}(0,1)$.

Show that $Y^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$.

- Set $Z = Y^2$: $\Rightarrow$ takes values in $\mathbb{R}^+$

$F_Z(x) = \mathbb{P}(Z \leq x) = \mathbb{P}(Y^2 \leq x) = \mathbb{P}(-\sqrt{x} \leq Y \leq \sqrt{x}) = F_Y(\sqrt{x}) - F_Y(-\sqrt{x})$

$= F_Y(x^{1/2}) - F_Y(-x^{1/2}) \Rightarrow f_Z(x) = \frac{d}{dx} F_Z(x)$

Hence $f_Z(x) = \frac{1}{2} x^{-1/2} f_Y(x^{1/2}) - \left(-\frac{1}{2} x^{-1/2} f_Y(-x^{1/2})\right)$

$$= \frac{1}{2} x^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x^{1/2})^2}{2}} + \frac{1}{2} x^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(-x^{1/2})^2}{2}}$$

$$= \frac{1}{2} x^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} + \frac{1}{2} x^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} = x^{-\frac{1}{2}} e^{-\frac{x}{2}} \frac{1}{\sqrt{2\pi}}$$

$$- \gamma_{\frac{1}{2},\frac{1}{2}}(x) = \frac{\sqrt{\frac{1}{2}}}{\Gamma(\frac{1}{2})} e^{-\frac{1}{2}x} x^{\frac{1}{2}-1} \mathbb{1}_{\mathbb{R}^+}(x) = \frac{1}{\Gamma(\frac{1}{2})\sqrt{2}} e^{-\frac{x}{2}} x^{-\frac{1}{2}} \mathbb{1}_{\mathbb{R}^+}(x)$$

Therefore $Y^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Exercice 18 Z is considered as a lognormal distribution with parameters m and σ $(\sigma > 0)$ if Z is a random variable which takes values in $\mathbb{R}^{+*}$ such that $\ln(Z) \sim \mathcal{N}(m, \sigma^2)$.

1- What is the density of Z ?

We know that $\ln(Z) = Y \sim \mathcal{N}(m, \sigma^2)$

Thus $Z = e^Y = \phi(Y)$, takes values in $\mathbb{R}^{+*}$

$F_Z(x) = \mathbb{P}(Z \leq x) = \mathbb{P}(e^Y \leq x) = \mathbb{P}(Y \leq \ln(x)) = F_Y(\ln(x))$

Hence $f_Z(x) = \frac{d}{dx} F_Z(x) = \frac{d}{dx} F_Y(\ln(x)) = \frac{1}{x} f_Y(\ln(x))$

Thus $f_Z(x) = \frac{1}{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(x)-m)^2}{2\sigma^2}} = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x - m)^2}{2\sigma^2}} \mathbb{1}_{\mathbb{R}^{+*}}(x)$

2. Give $\mathbb{E}[Z]$ and $Var(Z)$:

- $\mathbb{E}[Z] = \int_{-\infty}^{+\infty} x f_Z(x)\, dx = \int_{-\infty}^{+\infty} \cancel{x} \frac{1}{\cancel{x}\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x - m)^2}{2\sigma^2}} \mathbb{1}_{\mathbb{R}^{+*}}(x)\, dx$

$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x - m)^2}{2\sigma^2}}\, dx = \ldots$

Exercice 19

① What is the generating function of a Binomial $\mathcal{B}(n,p)$ r.v. ?

We remind that $G_X : [-1;1] \to \mathbb{R}$

$s \mapsto G_X(s) = \mathbb{E}[s^X]$ is defined for

random variables with values taken in $\mathbb{N}$. (which is the case here!)

Then, $G_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{n} s^k \,\mathbb{P}(X=k) = \sum_{k=0}^{n} s^k C_n^k p^k (1-p)^{n-k}$

$$= \sum_{k=0}^{n} C_n^k (sp)^k (1-p)^{n-k} \overset{\text{binom Newton}}{=} (sp+(1-p))^n = (1-p+sp)^n .$$

② - If $X \sim \mathcal{P}(\lambda)$, then

$$G_X(s) = \sum_{k\geq 0} s^k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k\geq 0} e^{-\lambda} \frac{(\lambda s)^k}{k!} = e^{-\lambda} \sum_{k\geq 0} \frac{(\lambda s)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda s} = e^{-\lambda(1-s)}$$

Exercice 20

1) Give the characteristic function of X when $X \sim \mathcal{E}xp(\lambda)$: $\lambda > 0$.

$$\phi_X(t) = \mathbb{E}[e^{itX}] = \int_{-\infty}^{+\infty} e^{itx} \lambda e^{-\lambda x} \mathbb{1}_{\{x>0\}} dx = \int_0^{+\infty} \lambda e^{(it-\lambda)x} dx$$

$$= \lambda \left[\frac{e^{(it-\lambda)x}}{it-\lambda} \right]_0^{+\infty}, \text{ where } e^{(it-\lambda)x} = e^{itx-\lambda x} = \frac{e^{itx}}{e^{\lambda x}}$$

complex number with module $=1$

$$= \lambda \left(\frac{1}{it-\lambda} \underbrace{\lim_{x\to+\infty} e^{-\lambda x}}_{=0} - \frac{1}{it-\lambda} \right) = \frac{\lambda}{\lambda - it}$$